A finite-dimensional model for affine, linear quantum lambda calculi with general recursion

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Abstract

We introduce a concrete domain model for the quantum lambda calculus and λ_ρ° extended with a fixpoint operator. A distinctive feature of λ_ρ° is that it relies on density matrices for describing both quantum information and probabilistic distributions over computation states. It has been shown that there is a conservative translation from λ_ρ° to the quantum lambda calculus of Selinger and Valiron. In contrast to existing models for quantum lambda calculi featuring recursion with intuitionistic arrows, our model is finite-dimensional and does not need more than cones of positive matrices and affine arrows.

2012 ACM Subject Classification Mathematics of computing → Lambda calculus; Theory of computation → Type theory; Theory of computation → Denotational semantics; Theory of computation → Quantum computation theory

Keywords and phrases Quantum lambda calculus, Denotational semantics, Fixpoint operator, Linear affine type system

Funding Funded by 21STIC10 Qapla'. *Díaz-Caro and Ivnisky*: Partially funded by PICT-2019-1272.

1 Introduction

Quantum computation is a model of computation where data is encoded on the state of particles governed by the laws of quantum physics. In the mathematical formalism, a piece of quantum data can be regarded as a complex linear combination of pieces of classical data: quantum states are represented with Hilbert spaces. To recover classical data from quantum information, the physical operation is called measurement: it is a *probabilistic* operation, modifying the global state of the system. The bottom line is that a quantum algorithm in general produces a *probabilistic distribution of pure quantum states*. The standard denotational semantics for quantum data consists in *density matrices*, i.e. *positive matrices of trace 1*. In this compact representation, eigenvectors correspond to classical outcomes while eigenvalues encompass the probability of getting each of them. In the historical interpretation [6], a quantum algorithm inputs a quantum state, operates on it and outputs the resulting modified state: this simple situation can be regarded as a *superoperator*, a trace-preserving linear map acting on positive matrices. The semantics of quantum algorithms in this approach is finite-dimensional: a quantum algorithm manipulates a finite amount of information.

The last twenty years have seen the development of quantum programming languages and semantics thereof. In particular, the design of functional programming languages for quantum computation has roots in the seminal work of Selinger [10], introducing *quantum flow charts* (QFCs). QFCs are (possibly recursive) first-order programs: the trace of the output might be smaller than 1, allowing the program to possibly diverge. The denotational semantics of QFCs therefore extends superoperators to *non-increasing* linear maps acting on cones of positive matrices. This approach has been subsequently extended in [11] to accommodate for higher-order programs: The *quantum lambda calculus* consists in a simply-typed, *linear* lambda calculus without recursion. Its denotational semantics is still finite-dimensional. It consists in an extension of the semantics of QFC where the requirement for trace-preservation for morphisms is relaxed. Objects are still cones of positive matrices, but morphisms are the so-called *completely positive maps* (CPM). Thanks to the compact-closure of the corresponding category [10, 12], this makes it possible to capture internal homs in this semantics.

If the CPM category can encode *linear* quantum higher-order computation, it is however limited in several ways. First, its finite dimensional aspect makes it impossible to account for duplicable data (as it would require the possibility to have non-linear functions) or inductive types. Then, although CPM-homsets can be endowed with a partial order consistent with the trace (the *Löwner order*), the lack of constant (non-zero) functions places the least fixed point of any function at 0, essentially saying that the fixpoint construction sending $A \to A$ to A is always diverging (as its probability is then 0).

The main problem that has been tackled in the literature [3, 4, 7, 8] consists in developing a semantics extending CPM to support infinite dimensional objects: the focus has been placed on duplicability [7] and inductive types [8]. In both cases, the extensions encompasses affine functions, allowing one to rely on the least fixed point construction to model recursion.

Contributions. In this work, we instead concentrate on the problem of designing a *finite dimensional* extension of the CPM denotational semantics supporting fixpoints. In particular, we do not need to account for duplicable objects nor inductive datatypes. To support our approach, we follow an operational approach by building up on a concrete quantum lambda-calculus featuring recursion while admitting a finite-dimensional model. Concretely, we extend the quantum lambda calculus λ_ρ° [5] with a fixpoint operator while forbidding duplicable elements. We remark that λ_ρ° has been shown equivalent [1] to the one proposed by Selinger&Valiron [11], but relies on a convenient presentation in terms of density matrices. The denotational semantics of our language follows the approach of Selinger [10]; consequently, we interpret basic types as positive matrices with trace less than or equal to 1. Following the usual intuition, a matrix whose trace is 1 represents a terminating program, while a matrix whose trace is smaller than 1 represents a program that might not terminate.

We build upon the standard Choi representation [2] of a completely positive linear map f as the positive matrix, and extend it to the affine case. We show that this finite model is sound and suffices to interpret the fixpoint operator as the least upper bound of a chain of approximations, i.e., as $\lim_{n\to\infty} \overline{\chi}_f^n(\mathbf{0})$ where $\overline{\chi}_f$ is the extended Choi representation of the affine function f, and $\mathbf{0}$ denotes the null matrix.

Plan of the paper. In Section 2 we introduce the calculus λ_ρ^μ and give some examples. In Section 3.1 we give a version of λ_ρ^μ with an incremental fixpoint operator parameterised by a bound on the number of iterations. The semantics interpretation of λ_ρ^μ and of this intermediate language are given in Sections 3.2 to 3.4. The adequacy for the intermediate language is proved in Section 4.1, which allows us to study the existence of the limit when

$$
\begin{aligned}
t ::= \quad & x \mid \lambda x.t \mid tt \mid \mu x.t && \textbf{(Standard lambda terms)}\\
& \mid \rho^n \mid U^n t \mid \pi^m t \mid t \otimes t && \textbf{(Quantum postulates)}\\
& \mid \sum_{i=1}^{n} p_i t_i \mid \mathsf{letcase}^{\circ}\ x = t \ \mathsf{in}\ \{t, \dots, t\} && \textbf{(Control)}
\end{aligned}
$$

where $\rho \in \mathcal{D}_n = \left\{\rho \mid \rho \in \mathbb{C}^{2^n \times 2^n} \text{ positive with } tr(\rho) \leq 1\right\}$,
for $i = 1, \dots, n,\ p_i > 0$
$\sum_{i=1}^{n} p_i \leq 1,$
$\sum$ is considered modulo associativity and commutativity

Figure 1 Syntax for the λ_ρ^μ calculus

the bound tends to infinity, which is proven in Section 4.2. We finally conclude in Section 5. Omitted technical material and proofs are provided in the appendices for the reviewers' convenience.

2 The calculus λ_ρ^μ

In this section, we introduce the calculus λ_ρ^μ, which extends λ_ρ° [5] with a fixpoint operator.

2.1 Syntax and Operational Semantics

The syntax of λ_ρ^μ calculus is given in Figure 1. Notation $\mathbb{C}^{n\times m}$ stands for the set of $(n \times m)$-dimensional matrices with coefficients in $\mathbb{C}$.

Terms are divided in three categories:

- Standard terms of the lambda calculus with fixpoint, namely, a variable x, an abstraction $\lambda x.t$, an application tr, and the fixpoint $\mu x.t$ of the abstraction $\lambda x.t$.
- Quantum postulates, which include a quantum state ρ^n, where ρ is an n-dimensional semidefinite positive Hermitian matrix (positive) with trace less than or equal to 1 (we shall also write σ and τ for matrices); the application $U^n t$ of the unitary operator U to the first n qubits of t; the measurement $\pi^m t$ of the first m qubits of t in the computational basis; and the tensor product of states $t \otimes r$.
- Control operators, where $\sum_{i=1}^{n} p_i t_i$ (also written $p_1 t_1 + \cdots + p_n t_n$) stands for the probabilistic superposition of the programs (or density matrices) t_i, each of them with probability p_i; and the term $\mathsf{letcase}^{\circ}\ x = r\ \mathsf{in}\ \{t_0, \dots, t_n\}$ that expresses the combination of the programs $t_0, \dots t_n$ according to a probability distribution given by the result of the measurement described by r.

We also fix the following set Val of values

$$\mathsf{Val} ::= \ \rho^n \mid \pi^m \rho^n \mid \sum_{i=1}^{n} p_i(\lambda x.t_i)$$

The operational semantics is presented in Figure 2. Reduction of lambda terms is the standard of the weak lambda calculus (that is, without reduction under lambda). The rule for $U^m \rho^n$ corresponds to the application of the unitary operator U over the density matrix ρ, where $\overline{U}$ stands for $U \otimes I_{n-m}$ with I_{m-n} being the $m - n$ dimensional identity matrix, and

(Standard lambda terms)

$$(\lambda x.t)r \longrightarrow t[x := r] \qquad\qquad \mu x.t \longrightarrow t[x := \mu x.t]$$

(Quantum postulates)

$$U^m \rho^n \longrightarrow (\overline{U}\rho\overline{U}^\dagger)^n \qquad\qquad \rho^n \otimes \sigma^m \longrightarrow (\rho \otimes \sigma)^{n \times m}$$

(Control)

$$\sum_i p_i \rho_i^n \longrightarrow (\sum_i p_i \rho_i)^n \qquad \sum_i (p_i t) \longrightarrow (\sum_i p_i) t \qquad (\sum_i p_i t_i) r \longrightarrow \sum_i p_i (t_i r)$$

$$\mathsf{letcase}^\circ\ x = \pi^m \rho^n \ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\} \longrightarrow \sum_{i=0}^{2^m-1} p_i t_i[x := \rho_i^n] \text{ with } \begin{cases} p_i = \mathsf{tr}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right) \\ \rho_i = \begin{cases} \frac{\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger}{p_i} & \text{if } p_i \neq 0 \\ \overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger & \text{if } p_i = 0 \end{cases} \end{cases}$$

(Contextual rules)

$$\frac{t \longrightarrow r}{ts \longrightarrow rs} \qquad \frac{t \longrightarrow r}{st \longrightarrow sr} \qquad \frac{t \longrightarrow r}{s \otimes t \longrightarrow s \otimes r} \qquad \frac{t \longrightarrow r}{U^m t \longrightarrow U^m r}$$

$$\frac{t \longrightarrow r}{\pi^m t \longrightarrow \pi^m r} \qquad \frac{t \longrightarrow r}{t \otimes s \longrightarrow r \otimes s} \qquad \frac{t_j \longrightarrow r}{\sum_{i=1}^n p_i t_i \longrightarrow p_j r + \sum_{\substack{i=1 \\ i \neq j}}^n p_i t_i}$$

$$\frac{t \longrightarrow r}{\mathsf{letcase}^\circ\ x = t\ \mathsf{in}\ \{s_0, \dots, s_{2^m-1}\} \longrightarrow \mathsf{letcase}^\circ\ x = r\ \mathsf{in}\ \{s_0, \dots, s_{2^m-1}\}}$$

Figure 2 Rewrite system for the λ_ρ^μ calculus

$\overline{U}^\dagger$ stands for the conjugate transpose of $\overline{U}$. The remaining rules for quantum postulates are self-explanatory.

The reduction rules for sums allow for their rewriting as density matrices. The evaluation of a term $\mathsf{letcase}^\circ\ x = \pi^m \rho^n\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}$ gives rise to the probabilistic combination of the branches $t_0, \dots, t_{2^m-1}$ where the probability assigned to each branch corresponds to the probability p_i of each of the possible outcomes ρ_i of the measurement $\pi^m \rho^n$. The bound variable x, which may appear on every t_i, is substituted with the corresponding measurement result ρ_i. The expression $\overline{|i\rangle\langle i|}$ in the side condition of this rule represents the projectors $|i\rangle\langle i| \otimes I_{m-n}$.

It is worth remarking that the operator π^m only measures the first m qubits of a density matrix. Should other set of qubits be measured, then the state can be previously transformed with the application of a suitable unitary operator that swaps the qubits as required.

The typing system for the λ_ρ^μ calculus is defined in Figure 3. The type n corresponds to density matrices of n-qubit states, while (m, n) (with $m \leq n$) stands for measurements over the first m qubits of n-qubit states. The arrow type $A \multimap B$ corresponds to affine functions from A to B.

Lambda terms are typed as in the affine lambda calculus, while typing rules for quantum postulates are straightforward. The typing rules for control terms use the auxiliary function on types $\ell(A)$, dubbed *last type of* A, which is inductively defined by:

$$\ell(n) = n \qquad \ell((m,n)) = (m,n) \qquad \ell(A \multimap B) = \ell(B)$$

Its usage is analogous in both rules (i.e., $+$ and m_e), where the premise $\ell(A) \neq (m, n)$ prevents the probabilistic combination of measurements. Note that π^m is the constructor for

$$A ::= n \mid (m,n) \mid A \multimap A \quad \text{where } m \leq n \in \mathbb{N}$$

(Standard lambda terms)

$$\frac{}{\Gamma, x : A \vdash x : A}\ \mathsf{ax} \qquad \frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \multimap_i \qquad \frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash r : A}{\Gamma, \Delta \vdash tr : B}\ \multimap_e$$

$$\frac{\Gamma, f : A \vdash t : A}{\Gamma \vdash \mu f.t : A}\ \mu$$

(Quantum postulates)

$$\frac{}{\Gamma \vdash \rho^n : n}\ \mathsf{ax}_\rho \qquad \frac{\Gamma \vdash t : n}{\Gamma \vdash U^m t : n}\ \mathsf{u}_i \qquad \frac{\Gamma \vdash t : n}{\Gamma \vdash \pi^m t : (m,n)}\ \mathsf{m}_i \qquad \frac{\Gamma \vdash t : n \quad \Delta \vdash r : m}{\Gamma, \Delta \vdash t \otimes r : n+m}\ \otimes$$

(Control)

$$\frac{{}^{i=1,\dots,n}\quad \Gamma \vdash t_i : A \quad \sum_{i=1}^n p_i \leq 1 \quad \ell(A) \neq (m,n)}{\Gamma \vdash \sum_{i=1}^n p_i t_i : A}\ +$$

$$\frac{{}^{i=0,\dots,2^m-1}\quad \Delta_i, x : n \vdash t_i : A \quad \Gamma \vdash r : (m,n) \quad \ell(A) \neq (m',n')}{\Delta_0, \dots, \Delta_{2^m-1}, \Gamma \vdash \mathsf{letcase}^\circ\ x = r \ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\} : A}\ \mathsf{m}_e$$

Figure 3 Typing system for the λ_ρ^μ calculus

measurements while the $\mathsf{letcase}^\circ$ construction is the destructor. Indeed, terms of the form $\pi^m t$ are only used inside their destructors $\mathsf{letcase}^\circ$.

Despite rule m_e allows x to be used in the different branches $t_0, \dots, t_{2^m-1}$, we remark that such duplication of variables does not violate the quantum no-cloning theorem because each branch corresponds to the continuation associated with a particular result of the measurement.

It has been shown that λ_ρ° (i.e., the fragment of λ_ρ^μ without fixpoints) enjoys progress of typed closed terms (i.e., any typed closed term is either a value or it reduces), strong normalisation, confluence, and subject reduction [5, 9].

▶ **Example 2.1** (Teleportation protocol)**.** The well-known teleportation protocol (see [6, Section 1.3.7]) can be implemented in λ_ρ^μ as follows. Let τ^1 be the unknown state to be teleported, and $\rho^2 = \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$ the first Bell state. Then, the teleportation protocol can be expressed as

$$\mathsf{letcase}^\circ\ x = \pi^2(H^1\ \mathit{CNOT}^2\ (\tau^1 \otimes \rho^2))\ \mathsf{in}\ \{x, Z_3^3 x, X_3^3 x, Z_3^3 X_3^3 x\}$$

where

- H is the Hadamard gate,
- CNOT is the Controlled-Not gate,
- Z is the Phase gate,
- X is the Not gate,
- $Z_3 = I_2 \otimes Z$,
- and $X_3 = I_2 \otimes X$.

This term has type 3, and reduces to a term σ^3, where σ is the 3-qubits state $I_2 \otimes \rho$.

▶ **Example 2.2.** We illustrate the definition of a recursive (non-terminating) term that converges to the state $|0\rangle\langle 0|$, which is defined as $\mathsf{F} = \mu x.\mathsf{letcase}^{\circ}\ z = \pi^1|+\rangle\langle+|\ \mathsf{in}\ \{x, |0\rangle\langle 0|\}$ and reduces as follows:

$$\begin{aligned}
\mathsf{F} = {} & \mu x.\mathsf{letcase}^{\circ}\ z = \pi^1|+\rangle\langle+|\ \mathsf{in}\ \{x, |0\rangle\langle 0|\} \\
& \longrightarrow\ \mathsf{letcase}^{\circ}\ z = \pi^1|+\rangle\langle+|\ \mathsf{in}\ \{\mathsf{F}, |0\rangle\langle 0|\} \\
& \longrightarrow\ \tfrac{1}{2}\mathsf{F} + \tfrac{1}{2}|0\rangle\langle 0| \\
& \longrightarrow\ \tfrac{1}{2}\mathsf{letcase}^{\circ}\ z = \pi^1|+\rangle\langle+|\ \mathsf{in}\ \{\mathsf{F}, |0\rangle\langle 0|\} + \tfrac{1}{2}|0\rangle\langle 0| \\
& \longrightarrow\ \tfrac{1}{2}\big(\tfrac{1}{2}\mathsf{F} + \tfrac{1}{2}|0\rangle\langle 0|\big) + \tfrac{1}{2}|0\rangle\langle 0| \\
& \longrightarrow^{*}\ \tfrac{1}{2}\Big(\tfrac{1}{2}\ldots\big(\tfrac{1}{2}\mathsf{F} + \tfrac{1}{2}|0\rangle\langle 0|\big)\ldots + \tfrac{1}{2}|0\rangle\langle 0|\Big) + \tfrac{1}{2}|0\rangle\langle 0| \\
& \longrightarrow\ \ldots
\end{aligned}$$

The term at Example 2.2 does not terminate, and grows infinitely. If, instead, at each coin toss (that is, the measurement π^1 over $|+\rangle\langle+|$), we would follow one of the paths with probability $\frac{1}{2}$, the global probability of no termination would be 0. Indeed, λ_ρ° internalises these paths in a sort of "generalised density matrix". In [5] there are two presentations of this calculus: λ_ρ, with a probabilistic rewrite system, and λ_ρ°, with the non-probabilistic rewrite system, as the one presented here, internalising the probabilistic paths. However, both presentations share the same semantics. In this paper we chose to follow λ_ρ° since it is closer to its semantics. Indeed, the denotation of the term at Example 2.2 will be shown to be the state $|0\rangle\langle 0|$, but such a denotation will be obtained as the limit of the iteration.

3 Denotational semantics on positive matrices

In this section we develop a denotational semantics of λ_ρ^μ, in which terms are interpreted as density matrices, along the lines of [10, 12]. From a semantic viewpoint, matrices with trace strictly smaller than 1 represent programs with a positive probability of non termination [10]. As customary, we rely on the *Löwner order* $\sqsubseteq$ over density matrices of dimension n defined such that $M \sqsubseteq N$ if and only if $M - N$ is a positive matrix. As shown in [10], density matrices of dimension n equipped with the Löwner order conforms a CPO that has the null matrix 0 as its least element. For functions, we adopt the CPM approach of [12]. However, our interpretation of functions allows us to accommodate affine maps, i.e., maps f such that $f(0) \neq 0$; this is achieved by representing each affine mapping as the composition of a linear transformation and a constant translation. This change is essential for the interpretation of terms $\mu x.t$ as the least fixed point of the denotation of $\lambda x.t$: if every abstraction were interpreted as a linear map, then its least fixed point would be also 0 (i.e., the bottom of the domain).

Technically, our definition of the denotational semantics of λ_ρ^μ is obtained indirectly from an intermediate calculus in which fixpoints are incremental, i.e., the fixpoint operator is parameterised by a natural number that bounds the possible iterations. The interpretation of fixpoints in λ_ρ^μ is obtained as the limit of the interpretation of the incremental fixpoint.

The remaining of this section is structured as follows. We start by extending λ_ρ^μ with incremental fixpoints in Section 3.1. In Section 3.2 we define domains and the interpretation of types. In Section 3.3, we give a canonical representation for affine maps as an extension of the classical Choi representation of CPMs. The interpretation of terms is presented in Section 3.4.

3.1 Incremental fixpoint

In order to account for incremental fixpoints, we extend the syntax of λ_ρ^μ as follows:

$$t ::= \dots \mu_n x.t \mid \bot$$

where fixpoint terms are labelled with a natural number n, and $\bot$ stands for the undefined value (which will be denoted by the null matrix) of the type A. The type system is extended by ignoring the labelling on the fixpoint, and adding the new axiom:

$$\frac{}{\Gamma \vdash \bot : A}$$

The rewrite system adds the following rules:

$$\begin{array}{rlrl} \mu_0 x.t & \longrightarrow \bot & \qquad pt + q\bot & \longrightarrow pt \\ \mu_{n+1} x.t & \longrightarrow t[x := \mu_n x.t] & \bot\ t & \longrightarrow \bot \end{array}$$

The rules for the fixpoint decrement n at each reduction until it reaches 0, when it reduces to $\bot$. The remaining two rules are instrumental to rearrange / propagate $\bot$.

Note that the incremental version of the term F in Example 2.2, i.e., $\mathsf{F}_n = \mu_n x.\mathsf{letcase}^\circ\ z = \pi^1 |+\rangle\langle+|\ \mathsf{in}\ \{x, |0\rangle\langle 0|\}$ rewrites to different values for every n and it converges to $|0\rangle\langle 0|$ when n approaches infinity. The matrix trace of the incremental results grows with n, from 0 when $n = 0$ to 1 when n tends to infinity. This trace can be seen as the probability for the program to converge, considering each measurement result as a probabilistic event.

3.2 Interpretation of Types

Recall that $\mathcal{D}_n$ is the set of density matrices of dimension 2^n.

Then, types are interpreted as follows:

$$\begin{aligned} (\!| n |\!) &= \mathcal{D}_n \\ (\!| (m,n) |\!) &= \left\{ M \mid M \in \bigoplus_{i=1}^{2^m} \mathcal{D}_n \text{ and } \mathsf{tr}\,(M) \leq 1 \right\} \\ (\!| A \multimap B |\!) &= \{ f \mid f \text{ positive in } ((\!| A |\!) \otimes (\!| B |\!)) \oplus (\!| B |\!) \} \end{aligned}$$

- The type n is interpreted as the set of density matrices of dimension 2^n, since they represent n-qubit systems.
- The type (m, n) is interpreted as the set of coproducts of 2^m density matrices of dimension 2^n, with global trace bounded by 1. Intuitively, the type (m, n) describes all the possible outcomes of measuring the first m qubits of a state of n qubits, i.e., the combination of 2^m possible states, each of them in $\mathcal{D}_n$. For example, $M = (\frac{1}{2}|0\rangle\langle 0| \oplus \frac{1}{2}|1\rangle\langle 1|) \in (\!|(1,1)|\!)$ because $\frac{1}{2}|0\rangle\langle 0| \in \mathcal{D}_1$, $\frac{1}{2}|1\rangle\langle 1| \in \mathcal{D}_1$ and $\mathsf{tr}\,(M) = 1$.
- The type $A \multimap B$ is interpreted as the set of positive matrices in $((\!|A|\!) \otimes (\!|B|\!)) \oplus (\!|B|\!)$, where the linear part is represented in $(\!|A|\!) \otimes (\!|B|\!)$ via its action on the canonical basis of $(\!|A|\!)$, and the constant part is a matrix in $(\!|B|\!)$.

▶ **Definition 3.1** (Domains). *The set* Dom *of interpretation domains is* $\mathsf{Dom} = \bigcup_{A \in \mathsf{Types}} (\!|A|\!)$.

▶ **Definition 3.2** (Dimension). *The dimension of a type is defined as the dimension of its representation space, that is* $\dim(A) = dim((\!|A|\!))$*:*

$$\dim(n) = 2^n \qquad \dim((m,n)) = 2^m 2^n = 2^{n+m} \qquad \dim(A \multimap B) = (\dim(A) + 1)\dim(B)$$

3.3 Extended Choi representation for affine functions

Affine functions consist of a linear transformation and a translation; consequently, we interpret a function $f : A \multimap B$ as matrix $\overline{\chi}_{[f]} \in (\!|A \multimap B|\!)$ that combines two matrices, one that represents its linear part and one that represents its constant part, i.e.,

$$\overline{\chi}_{[f]} = \left(\begin{array}{c|c|c} f(E^A_{11}) - f(\mathbf{0}_{\dim(A)}) & \dots & f(E^A_{1n}) - f(\mathbf{0}_{\dim(A)}) \\ \hline \vdots & \ddots & \vdots \\ \hline f(E^A_{n1}) - f(\mathbf{0}_{\dim(A)}) & \dots & f(E^A_{nn}) - f(\mathbf{0}_{\dim(A)}) \end{array}\right) \oplus f(\mathbf{0}_{\dim(A)})$$

where $\{E^A_{ij}\}$ are the elements of the canonical basis of $(\!|A|\!)$, and $\mathbf{0}_{\dim(A)}$ is the null matrix in $(\!|A|\!)$. The matrix on the left-hand-side of the coproduct represents the linear transformation on the canonical basis of $(\!|A|\!)$, and the right-hand-side matrix represents the translation.

We can also write this representation in terms of the characteristic matrix defined in [10, Section 6.7] for the linear function $f - f(\mathbf{0}_{\dim(A)})$. Let g be a linear function, its characteristic matrix is

$$\chi_{[g]} = \left(\begin{array}{c|c|c} g(E^A_{11}) & \dots & g(E^A_{1n}) \\ \hline \vdots & \ddots & \vdots \\ \hline g(E^A_{n1}) & \dots & g(E^A_{nn}) \end{array}\right)$$

Then $\overline{\chi}_{[f]}$ can equivalently be defined as $\overline{\chi}_{[f]} = \chi_{[f - f(\mathbf{0}_{\dim(A)})]} \oplus f(\mathbf{0}_{\dim(A)})$.

The application of an affine map to an element requires (i) decomposing the element in the canonical basis, (ii) applying the linear transformation to each individual component, and (ii) accumulate partial results also with the translation.

► **Definition 3.3** (Projection). *Let $\{E^n_{ij}\}$ be the canonical basis of the space $\mathbb{C}^{n\times n}$, and $\overline{\chi} = (\sum_{ij}(E^n_{ij} \otimes M_{ij})) \oplus M_\perp \in \mathbb{C}^{nm\times nm} \oplus \mathbb{C}^{m\times m}$. Then, the projection of $\overline{\chi}$ with respect to the indexes $1 \le k, l \le n$ is $P_{kl}(\overline{\chi}) = M_{kl}$. Moreover, $P_\perp(\overline{\chi}) = M_\perp$.*

Intuitively, the operator P_{kl} projects the submatrix $\mathbb{C}^{m\times m}$ corresponding to the linear component of $\overline{\chi}$ that corresponds to the basis E^n_{ij}, while $P_\perp$ projects the constant component of the mapping.

► **Definition 3.4** (Application). *Let $\overline{\chi} \in \mathbb{C}^{nm\times nm} \oplus \mathbb{C}^{m\times m}$. Then, the application of $\overline{\chi}$ to an element in $\mathbb{C}^{n\times n}$ is denoted by the operator $\#$, which is defined as follows:*

$$\overline{\chi} \# \Big(\sum_{ij} m_{ij} E^n_{ij}\Big) = \Big(\sum_{ij} m_{ij} P_{ij}(\overline{\chi})\Big) + P_\perp(\overline{\chi})$$

We shall write $\overline{\chi} \#_n M$ for n applications of $\overline{\chi}$ to M, e.g., $\overline{\chi} \#_3 M = \overline{\chi} \# (\overline{\chi} \# (\overline{\chi} \# M))$.

► Remark 3.5. The operator $\#$ can be defined in terms of the standard linear application @ of Choi matrices directly as $\overline{\chi}_{[f]} \# M = \Big(\chi_{[f-f(\mathbf{0}_n)]} \oplus f(\mathbf{0}_n)\Big) \# M = \chi_{[f-f(\mathbf{0}_n)]} @ M + f(\mathbf{0}_n)$.

► **Lemma 3.6** ($\#$ is right affine). *Let $\overline{\chi} \in \mathbb{C}^{nm\times nm} \oplus \mathbb{C}^{m\times m}$ and $M, N \in \mathbb{C}^{n\times n}$. Then, $\overline{\chi} \# (M + N) = \overline{\chi} \# M + \overline{\chi} \# N - P_\perp(\overline{\chi})$.* ◄

► **Lemma 3.7** ($\#$ is left linear). *Let $\overline{\chi}_1, \overline{\chi}_2 \in \mathbb{C}^{nm\times nm} \oplus \mathbb{C}^{m\times m}$ and $M \in \mathbb{C}^{n\times n}$. Then, $(\overline{\chi}_1 + \overline{\chi}_2) \# M = \overline{\chi}_1 \# M + \overline{\chi}_2 \# M$.* ◄

(Standard lambda terms)

$$
\begin{aligned}
(\!|x|\!)_\theta &= \theta(x) \\
(\!|\lambda x.t|\!)_\theta &= \overline{\chi}_{[a \mapsto (\!|t|\!)_{\theta,x=a}]} \\
(\!|tr|\!)_\theta &= (\!|t|\!)_\theta \ \# \ (\!|r|\!)_\theta \\
(\!|\mu x.t|\!)_\theta &= \lim_{n\to\infty} \left((\!|\lambda x.t|\!)_\theta \ \#_n \ \mathbf{0}_{\dim(A)} \right)
\end{aligned}
$$

(Quantum postulates)

$$
\begin{aligned}
(\!|\rho^n|\!)_\theta &= \rho \\
(\!|U^m t|\!)_\theta &= \overline{U} (\!|t|\!)_\theta \overline{U}^\dagger \\
(\!|\pi^m t|\!)_\theta &= \bigoplus_{i=0}^{2^m-1} \left(\overline{|i\rangle\langle i|} (\!|t|\!)_\theta \overline{|i\rangle\langle i|}^\dagger \right) \\
(\!|t \otimes r|\!)_\theta &= (\!|t|\!)_\theta \otimes (\!|r|\!)_\theta
\end{aligned}
$$

(Control)

$$
(\!|\sum_i p_i t_i|\!)_\theta = \sum_i p_i (\!|t_i|\!)_\theta
$$

$$
(\!|\mathsf{letcase}^\circ\ x = r \ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}|\!)_\theta = \sum_{i=0}^{2^m-1} \mathsf{tr}\,(\rho_i)\ (\!|t_i|\!)_{\theta, x=\rho'_i}
$$

$$
\text{with } (\!|r|\!)_\theta = \bigoplus_{i=0}^{2^m-1} \rho_i \text{ and } \rho'_i = \begin{cases} \frac{\rho_i}{\mathsf{tr}(\rho_i)} & \text{if } \mathsf{tr}\,(\rho_i) \neq 0 \\ \rho_i & \text{if } \mathsf{tr}\,(\rho_i) = 0 \end{cases}
$$

The type A is the type of the interpreted term.

Figure 4 Denotational semantics for the λ_ρ^μ calculus.

3.4 Interpretation of Terms

Our interpretation of terms depends on a *valuation function*, i.e., a partial function $\theta : \mathsf{Var} \to \mathsf{Dom}$ that maps each variable to an element of some domain. Then, the interpretation of a (typed) term t with respect to a valuation θ is inductively defined by the equations in Figure 4. For the sake of simplicity, we left implicit the typing judgement and write $(\!|t|\!)_\theta$ in lieu of $(\!|\Gamma \vdash t : A|\!)_\theta$. This is irrelevant for most of the equations but for abstractions and fixpoints. For abstractions, we rely on the representation of affine mappings introduced in Section 3.3, in this case, we implicitly assume $\Gamma \vdash \lambda x.t : A \multimap B$ and $a \in (\!|A|\!)$. Analogously, in the case of fixpoints we write $\mathbf{0}_{\dim(A)}$ to refer to the null matrix of a suitable dimension. In order to show that the interpretation function is well-defined, in particular, the existence of the limits in interpretation of fixpoints, we also provide an interpretation for incremental fixpoint by

$$
(\!|\mu_n x.t|\!)_\theta = (\!|\lambda x.t|\!)_\theta \ \#_n \ \mathbf{0}_{\dim(A)} \qquad\qquad (\!|\bot|\!)_\theta = \mathbf{0}_{\dim(A)}
$$

▶ **Example 3.8.** Consider the λ_ρ^μ term $\lambda x.t$ where $t = \mathsf{letcase}^\circ\ y = \pi^1 |+\rangle\langle+|$ in $\{x, |0\rangle\langle 0|\}$. Note that $\vdash \lambda x.t : 1 \multimap 1$. Then, $(\!|\lambda x.t|\!)_\emptyset = \overline{\chi}_{[f]}$ with $f = a \mapsto (\!|t|\!)_{x=a}$ for $a \in (\!|1|\!)$. We first note that the measurement in t is independent of a, i.e., $(\!|\pi^1|+\rangle\langle+||\!)_{x=a} = (\!|\pi^1|+\rangle\langle+||\!)_\emptyset$. By the interpretation of a measurement, $(\!|\pi^1|+\rangle\langle+||\!)_\emptyset = \frac{1}{2}|0\rangle\langle 0| \oplus \frac{1}{2}|1\rangle\langle 1|$. By the interpretation of letcase, $(\!|t|\!)_{x=a} = \frac{1}{2}(\!|x|\!)_{x=a} + \frac{1}{2}|0\rangle\langle 0| = \frac{1}{2}a + \frac{1}{2}|0\rangle\langle 0|$. Hence, $f = a \mapsto \frac{1}{2}a + \frac{1}{2}|0\rangle\langle 0|$. Consequently, $\overline{\chi}_{[f]} = \chi_{[a \mapsto \frac{1}{2}a]} \oplus \frac{1}{2}|0\rangle\langle 0|$, i.e.,

$$
\overline{\chi}_{[f]} = \begin{pmatrix} \frac{1}{2}|0\rangle\langle 0| & \frac{1}{2}|0\rangle\langle 1| \\ \frac{1}{2}|1\rangle\langle 0| & \frac{1}{2}|1\rangle\langle 1| \end{pmatrix} \oplus \frac{1}{2}|0\rangle\langle 0|
$$

Consider now the application $(\lambda x.t)\rho^1$ where $\rho = |0\rangle\langle 1| + |1\rangle\langle 1|$. Then, $(\!|(\lambda x.t)\rho^1|\!) = \overline{\chi}_{[f]} \ \# \ \rho = \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 1|)$.

▶ **Example 3.9.** Consider now the term $\lambda x.t$ with $t = \mathsf{letcase}^\circ\ y = x$ in $\{|0\rangle\langle 0|, |1\rangle\langle 1|\}$ and note that $\vdash \lambda x.t : (1,1) \multimap 1$. As in the previous example, $(\!|\lambda x.t|\!)_\emptyset = \overline{\chi}_{[g]}$ with $g = a \mapsto (\!|t|\!)_{x=a}$ for $a \in (\!|(1,1)|\!)$. The canonical basis of the domain $(\!|(1,1)|\!)$ has the following elements:

$$
\begin{array}{cccc}
|0\rangle\!\langle 0| \oplus \mathbf{0}_2 & |1\rangle\!\langle 0| \oplus \mathbf{0}_2 & \mathbf{0}_2 \oplus |0\rangle\!\langle 1| & \mathbf{0}_2 \oplus |1\rangle\!\langle 1| \\
|0\rangle\!\langle 1| \oplus \mathbf{0}_2 & |1\rangle\!\langle 1| \oplus \mathbf{0}_2 & \mathbf{0}_2 \oplus |0\rangle\!\langle 0| & \mathbf{0}_2 \oplus |1\rangle\!\langle 0|
\end{array}
$$

Then, $\overline{\chi}_{[g]}$ is given by

$$\overline{\chi}_{[g]} = \left(\begin{pmatrix} |0\rangle\langle 0| & \mathbf{0}_2 \\ \mathbf{0}_2 & |0\rangle\langle 0| \end{pmatrix} \oplus \begin{pmatrix} |1\rangle\langle 1| & \mathbf{0}_2 \\ \mathbf{0}_2 & |1\rangle\langle 1| \end{pmatrix} \right) \oplus \mathbf{0}_2$$

▶ **Example 3.10.** Consider the λ_ρ^μ term in Example 2.2. Its interpretation is:

$$
\begin{aligned}
&(\!|\mu x.\mathsf{letcase}^\circ\ z = \pi^1 |+\rangle\!\langle +| \text{ in } \{x, |0\rangle\!\langle 0|\}|\!)_\emptyset \\
&= \lim_{n\to\infty} (\!|\lambda x.\mathsf{letcase}^\circ\ z = \pi^1 |+\rangle\!\langle +| \text{ in } \{x, |0\rangle\!\langle 0|\}|\!)_\emptyset \#_n\ \mathbf{0}_2
\end{aligned}
$$

Note that $(\!|\lambda x.\mathsf{letcase}^\circ\ z = \pi^1 |+\rangle\!\langle +| \text{ in } \{x, |0\rangle\!\langle 0|\}|\!)_\emptyset$ is $\overline{\chi}_{[f]}$ defined in Example 3.8. Assuming that the limit exists, we have

$$
\begin{aligned}
\lim_{n\to\infty} \overline{\chi}_{[f]} \#_n\ \mathbf{0}_2 &= \lim_{n\to\infty} \tfrac{1}{2^n}\mathbf{0}_2 + \sum_{i=1}^{n} \tfrac{1}{2^i} |0\rangle\!\langle 0| \\
&= \lim_{n\to\infty} \left(\frac{1-\frac{1}{2^{n+1}}}{1-\frac{1}{2}} - 1 \right) |0\rangle\!\langle 0| \\
&= |0\rangle\!\langle 0|
\end{aligned}
$$

▶ **Example 3.11.** The identity function has infinite fixpoints, all elements in the domain. Its fixpoint is defined as the least element in the domain, i.e., the null matrix:

$$(\!|\mu x.x|\!)_\emptyset = \lim_{n\to\infty} (\!|\lambda x.x|\!)_\emptyset \#_n\ \mathbf{0}_2 = \mathbf{0}_2$$

4 Adequacy of the interpretation

In this section we prove that our definition of interpretation is adequate, i.e., the interpretation function maps a term of type A to an element of the domain of the type A. We first address this problem for the intermediate language with incremental fixpoints (Section 4.1). In Section 4.2 we address the λ_ρ^μ calculus and show that the interpretation of fixpoint is well-defined, i.e., limits exist. This is achieved by showing that the interpretation function maps well-typed terms into matrices whose traces are bounded by the types of the terms. Such bounds allow us to show that the proposed domains equipped with the Löwner order are CPOs.

4.1 Adequacy of the incremental fixpoint calculus

We start by introducing some auxiliary notions and results that are instrumental for proving the adequacy of the interpretation function. Omitted proofs are given in Appendix B.

We say that a valuation θ and a typing context Γ are consistent, written $\theta \vDash \Gamma$, if and only if for every $x : A \in \Gamma$ we have $\theta(x) \in (\!|A|\!)$.

▶ **Lemma 4.1** (Linearity). *If $\Gamma, x : A \vdash t : B$ and $\theta \vDash \Gamma$, then for all $a \in (\!|A|\!)$ the function $a \mapsto (\!|t|\!)_{\theta,x=a} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}$ is linear.*

The following lemma states that the application operator behaves as expected with respect to the valuation used for the interpretation of terms.

▶ **Lemma 4.2** (Application). *If $\Gamma, x : A \vdash t : B$ and $\theta \vDash \Gamma$, then for all $a \in \mathbb{C}^{\dim(A)\times\dim(A)}$ we have $(\!|\lambda x.t|\!)_\theta \mathbin{\#} a = (\!|t|\!)_{\theta,x=a}$.*

The following is the usual expected substitution lemma.

▶ **Lemma 4.3** (Substitution). *$(\!|t[x := r]|\!)_\theta = (\!|t|\!)_{\theta,x=(\!|r|\!)_\theta}$.*

Next lemma shows that interpretation is stable with respect to reduction.

▶ **Lemma 4.4** (Reduction correctness). *If $\Gamma \vdash t : A$, $\theta \vDash \Gamma$, and $t \longrightarrow r$ then $(\!|t|\!)_\theta = (\!|r|\!)_\theta$.*

The proof of the main result in this section (Theorem 4.9) relies on auxiliary lemmas that state the adequacy for arrow types and sums. The case for abstractions is indirectly obtained from Lemma 4.5, which states that the linear part of the interpretation of an abstraction (see Lemma 4.1) is a completely positive map (CPM). This particularly means that when applied on positive matrices return positive matrices, so they belong to the domain.

▶ **Lemma 4.5.** *Let $\Gamma, x : A \vdash t : B$, and for all $\theta \vDash \Gamma$ and $a \in (\!|A|\!)$, let $(\!|t|\!)_{\theta,x=a} \in (\!|B|\!)$. Then, the map $F^{t,x}_{\theta\vDash\Gamma} = a \mapsto (\!|t|\!)_{\theta,x=a} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}$ is a CPM.*

▶ **Lemma 4.6** (Adequacy for abstractions). *Let $\Gamma, x : A \vdash t : B$ and $\theta \vDash \Gamma$, such that $(\!|t|\!)_{\theta,x=a}, (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} \in (\!|B|\!)$. Then $\overline{\chi}_{[a\mapsto(\!|t|\!)_{\theta,x=a}]} \in (\!|A \multimap B|\!)$.*

The previous lemma gives the adequacy of abstractions. Lemma 4.7 extends this result for all arrow-type terms.

▶ **Lemma 4.7** (Adequacy for arrow-type terms). *Let $\Gamma \vdash t : A \multimap B$ and $\theta \vDash \Gamma$. One of the following holds:*

- *There exist $t_1, \dots, t_n$ and $p_1, \dots, p_n$ such that $x : A \vdash t_i : B$, $p_i > 0$, $\sum_{i=1}^n p_i \leq 1$ and $(\!|t|\!)_\theta = \sum_{i=1}^n p_i (\!|\lambda x.t_i|\!)_\emptyset$.*
- *$(\!|t|\!)_\theta = \mathbf{0}_{\dim(A\multimap B)}$*

The adequacy for sums is stated below.

▶ **Lemma 4.8.** *Let $(\!|t_i|\!)_\theta \in (\!|A|\!)$ for $i \in \{1, \dots, n\}$. Then for any $p_1, \dots, p_n$ such that $0 < p_i \leq 1$ with $\sum_{i=1}^n p_i \leq 1$, we have $\sum_{i=1}^n p_i (\!|t_i|\!)_\theta \in (\!|A|\!)$.*

Finally we state the adequacy theorem.

▶ **Theorem 4.9** (Adequacy). *Let $\Gamma \vdash t : A$ and $\theta \vDash \Gamma$, then $(\!|t|\!)_\theta \in (\!|A|\!)$.*

4.2 Existence of fixpoints

We now proceed to show that $\lim_{n\to\infty}((\!|\lambda x.t|\!)_\theta \#_n \mathbf{0}_{\dim(A)})$ actually exists for well-typed terms (omitted proofs are in Appendix C). Firstly, we show that closed terms are interpreted as matrices with bounded traces. For terms of type n or (m, n), their trace is bounded by 1 by definition. Contrastingly, the bound is not directly given in the definition of the domains associated with arrow types. Since arrows $A \multimap B$ are interpreted in $((\!|A|\!) \otimes (\!|B|\!)) \oplus (\!|B|\!)$, their traces can be greater than 1. For example, the identity function in $1 \multimap 1$ has an interpretation of trace 2:

$$(\!|\lambda x.x|\!)_\emptyset = \overline{\chi}_{[a\mapsto(\!|x|\!)_{x=a}]} = (|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|) \oplus \mathbf{0}_2$$

However, we can associate a bound to a type, which we call its size.

▶ **Definition 4.10** (Size of a type). *Let A be a type, we write N_A for its size, which is inductively defined as follows:*

- $N_n = 1$
- $N_{(m,n)} = 1$
- $N_{A\multimap B} = (\dim(A) + 1)N_B$

▶ **Theorem 4.11.** *Let $\vdash t\colon A$, then* $\mathsf{tr}\,(⦇t⦈_\emptyset) \leq N_A$.

Next, we define the Löwner order (Definition 4.12) and show that abstractions terms in the calculus with incremental fixpoint that have identity type preserve this order (Lemma 4.13) and are continuous (Lemma 4.14).

▶ **Definition 4.12** (Löwner order). *Let M, N be positive matrices. Then $M \sqsubseteq N$ if and only if $N - M$ is positive.*

▶ **Lemma 4.13.** *Let $\Gamma, x : A \vdash t : A$ and $\theta \vDash \Gamma$. Then for all $a, b \in ⦇A⦈$, if $a \sqsubseteq b$ we have $⦇\lambda x.t⦈_\theta \# a \sqsubseteq ⦇\lambda x.t⦈_\theta \# b$.*

▶ **Lemma 4.14.** *Let $\overline{\chi} \in \mathbb{C}^{nm\times nm} \oplus \mathbb{C}^{m\times m}$ and let (P_n) be an increasing sequence of positive matrices in $\mathbb{C}^{n\times n}$ such that $\lim_{n\to\infty} P_n = P$. Then, $\overline{\chi}$ is monotone and $\lim_{n\to\infty} \overline{\chi} \# P_n = \overline{\chi} \# P$.*

Remark that for the case of square complex matrices, the least upper bound of an increasing sequence is equal to its limit (see [10, Remark 3.8]).

We have already shown that every term in the calculus with incremental fixpoint is interpreted as a positive matrix (Theorem 4.9) whose trace is bounded by the size of its type (Theorem 4.11). Moreover, we have shown that terms of type $A \multimap A$ are interpreted as continuous functions. This allows us to show that the image of the interpretations form CPOs.

For any type A, we let $\mathfrak{D}_A = \{M \mid M \in ⦇A⦈ \text{ and } \mathsf{tr}\,(M) \leq N_A\}$. Notice that, by Theorem 4.11, the image of the interpretation of closed terms $\vdash t : A$ lies on $\mathfrak{D}_A$. The following lemma states that this set, with the Löwner order, forms a CPO.

▶ **Lemma 4.15.** *For any type A, $(\mathfrak{D}_A, \sqsubseteq)$ is a complete partial order.*

Finally, the following theorem states that the denotation of the fixpoint in λ_ρ^μ is well defined.

▶ **Theorem 4.16.** *If $\Gamma \vdash \lambda x.t : A \multimap A$ and $\theta \vDash \Gamma$, then $\lim_{n\to\infty}(⦇\lambda x.t⦈_\theta \#_n \mathbf{0}_{\dim(A)}) \in \mathfrak{D}_A$.*

5 Conclusion

In this paper, we presented a finite-dimensional semantics for quantum higher-order computation with recursion. On one hand, to stay finite-dimensional we only account for non-duplicable elements. On the other hand, to be able to represent general recursion we allow the discarding of variables: the model is thus affine. In particular, we show how to extend the Choi-construction to linear, affine maps. The model is justified by the λ-calculus λ_ρ°, providing a concrete operational account of what affine, linear higher-order computation represents.

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A Fixpoint in CPOs

▶ **Definition A.1** (Complete partial order [13, p. 70]). *Let $(D, \preccurlyeq)$ be a partial order.*

An ω-chain of the partial order is an increasing chain $d_0 \preccurlyeq d_1 \preccurlyeq \cdots \preccurlyeq d_n \preccurlyeq \dots$ of elements of the partial order.

The partial order $(D, \preccurlyeq)$ is a complete partial order (abbreviated to CPO) if it has least upper bounds of all ω-chains.

We say $(D, \preccurlyeq)$ is a CPO with bottom if it is a CPO which has a least element $\bot$ (called "bottom").

▶ **Definition A.2** (Continuous function [13, p. 71]). *A function $f : (D, \preccurlyeq_D) \to (E, \preccurlyeq_E)$ between CPOs $(D, \preccurlyeq_D)$ and $(E, \preccurlyeq_E)$ is monotonic if and only if*

$$\forall d, d' \in D, d \preccurlyeq_D d' \implies f(d) \preccurlyeq_E f(d')$$

Such a function is continuous if and only if it is monotonic and for all chains $d_0 \preccurlyeq_D d_1 \preccurlyeq_D \ldots \preccurlyeq_D d_n \preccurlyeq_D \ldots$ *in* D *we have*

$$\bigsqcup_{n\in\omega} f(d_n) = f(\bigsqcup_{n\in\omega} d_n)$$

▶ **Theorem A.3** (Fixpoint theorem [13, Theorem 5.11]). *Let* $f : D \to D$ *be a continuous function on a CPO with bottom* D. *Define*

$$fix(f) = \bigsqcup_{n\in\omega} f^n(\bot)$$

Then $fix(f)$ *is the least fixpoint of* f.

B Proofs of Section 4.1

B.1 Proof of Lemma 4.1

▶ **Lemma 4.1** (Linearity). *If* $\Gamma, x : A \vdash t : B$ *and* $\theta \vDash \Gamma$, *then for all* $a \in (\!|A|\!)$ *the function* $a \mapsto (\!|t|\!)_{\theta,x=a} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}$ *is linear.*

Proof. Let $n = \dim(A)$. The defined function is linear if:

$$(\!|t|\!)_{\theta,x=\alpha A+\beta B} - (\!|t|\!)_{\theta,x=\mathbf{0}_n} = \alpha((\!|t|\!)_{\theta,x=A} - (\!|t|\!)_{\theta,x=\mathbf{0}_n}) + \beta((\!|t|\!)_{\theta,x=B} - (\!|t|\!)_{\theta,x=\mathbf{0}_n})$$

The linearity condition then is equivalent to:

$$(\!|t|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|t|\!)_{\theta,x=A} + \beta(\!|t|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|t|\!)_{\theta,x=\mathbf{0}_n} \tag{1}$$

By induction on t:

- Let $t = x$.

$$\begin{aligned}(\!|x|\!)_{\theta,x=\alpha A+\beta B} &= \alpha A + \beta B\\ &= \alpha A + \beta B - (\alpha+\beta-1)\ \mathbf{0}_n\\ &= \alpha(\!|x|\!)_{\theta,x=A} + \beta(\!|x|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|x|\!)_{\theta,x=\mathbf{0}_n}\end{aligned}$$

- Let $t = y \neq x$. On the left side we have:

$$(\!|y|\!)_{\theta,x=\alpha A+\beta B} = (\!|y|\!)_\theta = \theta(y)$$

  On the right side:

$$\alpha(\!|y|\!)_{\theta,x=A} + \beta(\!|y|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|y|\!)_{\theta,x=\mathbf{0}_n} = \alpha\theta(y) + \beta\theta(y) - (\alpha+\beta-1)\theta(y) = \theta(y)$$

- Let $t = \lambda y.r$. We want to show that:

$$(\!|\lambda y.r|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|\lambda y.r|\!)_{\theta,x=A} + \beta(\!|\lambda y.r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|\lambda y.r|\!)_{\theta,x=\mathbf{0}_n}$$

  By $(\!|\cdot|\!)_\theta$ definition this is equivalent to showing that:

$$\overline{\chi}_{[a\mapsto(\!|r|\!)_{\theta,x=\alpha A+\beta B,y=a}]} = \alpha\overline{\chi}_{[a\mapsto(\!|r|\!)_{\theta,x=A,y=a}]} + \beta\overline{\chi}_{[a\mapsto(\!|r|\!)_{\theta,x=B,y=a}]} - (\alpha+\beta-1)\overline{\chi}_{[a\mapsto(\!|r|\!)_{\theta,x=\mathbf{0}_n,y=a}]} \tag{2}$$

  Rewriting the left-hand side, by inductive hypothesis we have:

$$\overline{\chi}_{[a\mapsto(\!|r|\!)_{\theta,x=\alpha A+\beta B,y=a}]} = \overline{\chi}_{[a\mapsto\alpha(\!|r|\!)_{\theta,x=A,y=a}+\beta(\!|r|\!)_{\theta,x=B,y=a}-(\alpha+\beta-1)(\!|r|\!)_{\theta,x=\mathbf{0}_n,y=a}]}$$

  This is equal to the right-hand side of Equation 2, by linearity in matrix addition and matrix product with scalars.

- Let $t = rs$. Replacing in Equation 1, we want to show that:

$$(\!|rs|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|rs|\!)_{\theta,x=A} + \beta(\!|rs|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|rs|\!)_{\theta,x=\mathbf{0}_n} \tag{3}$$

Since the type system is affine, either $x \in \mathsf{FV}(r)$ or $x \in \mathsf{FV}(s)$.

  - If $x \in \mathsf{FV}(r)$, from Equation 3 we want to show that:

$$\begin{aligned}&(\!|r|\!)_{\theta,x=\alpha A+\beta B} \# (\!|s|\!)_\theta = \\ &\qquad \alpha((\!|r|\!)_{\theta,x=A} \# (\!|s|\!)_\theta) + \beta((\!|r|\!)_{\theta,x=B} \# (\!|s|\!)_\theta) - (\alpha+\beta-1)((\!|r|\!)_{\theta,x=\mathbf{0}_n} \# (\!|s|\!)_\theta)\end{aligned} \tag{4}$$

  Rewriting the left-hand side we have:

$$\begin{aligned}(\!|r|\!)_{\theta,x=\alpha A+\beta B} \# (\!|s|\!)_\theta &\overset{IH}{=} (\alpha(\!|r|\!)_{\theta,x=A} + \beta(\!|r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|r|\!)_{\theta,x=\mathbf{0}_n}) \# (\!|s|\!)_\theta \\ &\overset{(3.7)}{=} \alpha((\!|r|\!)_{\theta,x=A} \# (\!|s|\!)_\theta) + \beta((\!|r|\!)_{\theta,x=B} \# (\!|s|\!)_\theta) - (\alpha+\beta-1)((\!|r|\!)_{\theta,x=\mathbf{0}_n} \# (\!|s|\!)_\theta)\end{aligned}$$

  We obtain Equation 4's right-hand side.

  - If $x \in \mathsf{FV}(s)$, from Equation 3 we want to show that:

$$\begin{aligned}&(\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=\alpha A+\beta B} = \\ &\quad \alpha((\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=A}) + \beta((\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=B}) - (\alpha+\beta-1)((\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=\mathbf{0}_n})\end{aligned} \tag{5}$$

  Rewriting the left-hand side we have:

$$\begin{aligned}(\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=\alpha A+\beta B} &\overset{IH}{=} (\!|r|\!)_\theta \# (\alpha(\!|s|\!)_{\theta,x=A} + (\beta(\!|s|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|s|\!)_{\theta,x=\mathbf{0}_n})) \\ &\overset{(3.6)}{=} (\!|r|\!)_\theta \# (\alpha(\!|s|\!)_{\theta,x=A}) + (\!|r|\!)_\theta \# (\beta(\!|s|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|s|\!)_{\theta,x=\mathbf{0}_n}) - ((\!|r|\!)_\theta \# \mathbf{0}_n) \\ &\overset{(3.6)}{=} (\!|r|\!)_\theta \# (\alpha(\!|s|\!)_{\theta,x=A}) + (\!|r|\!)_\theta \# (\beta(\!|s|\!)_{\theta,x=B}) + (\!|r|\!)_\theta \# (-(\alpha+\beta-1)(\!|s|\!)_{\theta,x=\mathbf{0}_n}) - 2((\!|r|\!)_\theta \# \mathbf{0}_n) \\ &\overset{(3.6)}{=} \alpha((\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=A}) + (1-\alpha)((\!|r|\!)_\theta \# \mathbf{0}_n) + \beta((\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=B}) + (1-\beta)((\!|r|\!)_\theta \# \mathbf{0}_n) \\ &\quad - (\alpha+\beta-1)((\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=\mathbf{0}_n}) + (\alpha+\beta)((\!|r|\!)_\theta \# \mathbf{0}_n) - 2((\!|r|\!)_\theta \# \mathbf{0}_n) \\ &= \alpha((\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=A}) + \beta((\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=B}) - (\alpha+\beta-1)((\!|r|\!)_\theta \# (\!|s|\!)_{\theta,x=\mathbf{0}_n})\end{aligned}$$

  We obtain Equation 5's right-hand side.

- Let $t = \mu_m y.r$. In this case we want to show that, replacing in Equation 1:

$$(\!|\mu_m y.r|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|\mu_m y.r|\!)_{\theta,x=A} + \beta(\!|\mu_m y.r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|\mu_m y.r|\!)_{\theta,x=\mathbf{0}_n}$$

Rewriting the left-hand side:

$$\begin{aligned}(\!|\mu_m y.r|\!)_{\theta,x=\alpha A+\beta B} &= ((\!|\lambda y.r|\!)_{\theta,x=\alpha A+\beta B}) \#_m \mathbf{0}_n \\ &\overset{abs}{=} (\alpha(\!|\lambda y.r|\!)_{\theta,x=A} + \beta(\!|\lambda y.r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|\lambda y.r|\!)_{\theta,x=\mathbf{0}_n}) \#_m \mathbf{0}_n \\ &\overset{(3.7)}{=} \alpha\,((\!|\lambda y.r|\!)_{\theta,x=A}) \#_m \mathbf{0}_n + \beta\,((\!|\lambda y.r|\!)_{\theta,x=B}) \#_m \mathbf{0}_n - (\alpha+\beta-1)\,((\!|\lambda y.r|\!)_{\theta,x=\mathbf{0}_n}) \#_m \mathbf{0}_n \\ &= \alpha(\!|\mu_m y.r|\!)_{\theta,x=A} + \beta(\!|\mu_m y.r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|\mu_m y.r|\!)_{\theta,x=\mathbf{0}_n}\end{aligned}$$

- Let $t = \bot$. This case holds trivially since $(\!|\bot|\!)_\theta = \mathbf{0}_{\dim(A)}$ for every θ.
- Let $t = \rho^n$. This case is analogous to $t = y \neq x$.

- Let $t = U^m r$. Replacing in Equation 1 we want to show that:

$$(\!|U^m r|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|U^m r|\!)_{\theta,x=A} + \beta(\!|U^m r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|U^m r|\!)_{\theta,x=\mathbf{0}_n}$$

Rewriting the left-hand side:

$$\begin{aligned}(\!|U^m r|\!)_{\theta,x=\alpha A+\beta B} &= \overline{U}(\!|r|\!)_{\theta,x=\alpha A+\beta B}\overline{U}^\dagger \\ &\overset{IH}{=} \overline{U}(\alpha(\!|r|\!)_{\theta,x=A} + \beta(\!|r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|r|\!)_{\theta,x=\mathbf{0}_n})\overline{U}^\dagger \\ &= \alpha(\overline{U}(\!|r|\!)_{\theta,x=A}\overline{U}^\dagger) + \beta(\overline{U}(\!|r|\!)_{\theta,x=B}\overline{U}^\dagger) - (\alpha+\beta-1)(\overline{U}(\!|r|\!)_{\theta,x=\mathbf{0}_n}\overline{U}^\dagger) \\ &= \alpha(\!|U^m r|\!)_{\theta,x=A} + \beta(\!|U^m r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|U^m r|\!)_{\theta,x=\mathbf{0}_n}\end{aligned}$$

- Let $t = \pi^m r$. In this case we want to show that, replacing in Equation 1:

$$(\!|\pi^m r|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|\pi^m r|\!)_{\theta,x=A} + \beta(\!|\pi^m r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|\pi^m r|\!)_{\theta,x=\mathbf{0}_n}$$

Rewriting the left-hand side:

$$\begin{aligned}(\!|\pi^m r|\!)_{\theta,x=\alpha A+\beta B} &= \bigoplus_{i=1}^{2^m}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=\alpha A+\beta B}\overline{|i\rangle\langle i|}^\dagger\right) \\ &\overset{IH}{=} \bigoplus_{i=1}^{2^m}\left(\overline{|i\rangle\langle i|}(\alpha(\!|r|\!)_{\theta,x=A} + \beta(\!|r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|r|\!)_{\theta,x=\mathbf{0}_n})\overline{|i\rangle\langle i|}^\dagger\right) \\ &= \bigoplus_{i=1}^{2^m}\left(\alpha\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=A}\overline{|i\rangle\langle i|}^\dagger + \beta\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=B}\overline{|i\rangle\langle i|}^\dagger - (\alpha+\beta-1)\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=\mathbf{0}_n}\overline{|i\rangle\langle i|}^\dagger\right) \\ &= \alpha\bigoplus_{i=1}^{2^m}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=A}\overline{|i\rangle\langle i|}^\dagger\right) + \beta\bigoplus_{i=1}^{2^m}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=B}\overline{|i\rangle\langle i|}^\dagger\right) - (\alpha+\beta-1)\bigoplus_{i=1}^{2^m}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=\mathbf{0}_n}\overline{|i\rangle\langle i|}^\dagger\right) \\ &= \alpha(\!|\pi^m r|\!)_{\theta,x=A} + \beta(\!|\pi^m r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|\pi^m r|\!)_{\theta,x=\mathbf{0}_n}\end{aligned}$$

- Let $t = r \otimes s$. In this case we want to show that, replacing in Equation 1:

$$(\!|r\otimes s|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|r\otimes s|\!)_{\theta,x=A} + \beta(\!|r\otimes s|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|r\otimes s|\!)_{\theta,x=\mathbf{0}_n} \tag{6}$$

Since the type system is affine, either $x \in \mathsf{FV}(r)$ or $x \in \mathsf{FV}(s)$ holds.

  - If $x \in \mathsf{FV}(r)$, from Equation 6 we want to show that:

$$(\!|r|\!)_{\theta,x=\alpha A+\beta B}\otimes(\!|s|\!)_\theta = \alpha(\!|r|\!)_{\theta,x=A}\otimes(\!|s|\!)_\theta + \beta(\!|r|\!)_{\theta,x=B}\otimes(\!|s|\!)_\theta - (\alpha+\beta-1)(\!|r|\!)_{\theta,x=\mathbf{0}_n}\otimes(\!|s|\!)_\theta$$

  Rewriting the left-hand side:

$$\begin{aligned}(\!|r|\!)_{\theta,x=\alpha A+\beta B}\otimes(\!|s|\!)_\theta &\overset{IH}{=} (\alpha(\!|r|\!)_{\theta,x=A} + \beta(\!|r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|r|\!)_{\theta,x=\mathbf{0}_n})\otimes(\!|s|\!)_\theta \\ &= \alpha(\!|r|\!)_{\theta,x=A}\otimes(\!|s|\!)_\theta + \beta(\!|r|\!)_{\theta,x=B}\otimes(\!|s|\!)_\theta - (\alpha+\beta-1)(\!|r|\!)_{\theta,x=\mathbf{0}_n}\otimes(\!|s|\!)_\theta\end{aligned}$$

  - If $x \in \mathsf{FV}(s)$, from Equation 6 we want to show that:

$$(\!|r|\!)_\theta\otimes(\!|s|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|r|\!)_\theta\otimes(\!|s|\!)_{\theta,x=A} + \beta(\!|r|\!)_\theta\otimes(\!|s|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|r|\!)_\theta\otimes(\!|s|\!)_{\theta,x=\mathbf{0}_n}$$

  Rewriting the left-hand side:

$$\begin{aligned}(\!|r|\!)_\theta\otimes(\!|s|\!)_{\theta,x=\alpha A+\beta B} &\overset{IH}{=} (\!|r|\!)_\theta\otimes(\alpha(\!|s|\!)_{\theta,x=A} + \beta(\!|s|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|s|\!)_{\theta,x=\mathbf{0}_n}) \\ &= \alpha(\!|r|\!)_\theta\otimes(\!|s|\!)_{\theta,x=A} + \beta(\!|r|\!)_\theta\otimes(\!|s|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|r|\!)_\theta\otimes(\!|s|\!)_{\theta,x=\mathbf{0}_n}\end{aligned}$$

- Let $t = \sum_i p_i t_i$, with $\sum_i p_i \leq 1$ and $0 < p_i \leq 1$ for all i. In this case we want to show that, replacing in Equation 1:

$$(\!|\sum_i p_i t_i|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|\sum_i p_i t_i|\!)_{\theta,x=A} + \beta(\!|\sum_i p_i t_i|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|\sum_i p_i t_i|\!)_{\theta,x=\mathbf{0}_n}$$

Rewriting the left-hand side:

$$\begin{aligned}(\!|\sum_i p_i t_i|\!)_{\theta,x=\alpha A+\beta B} &= \sum_i p_i (\!|t_i|\!)_{\theta,x=\alpha A+\beta B} \\ &\overset{IH}{=} \sum_i p_i(\alpha(\!|t_i|\!)_{\theta,x=A} + \beta(\!|t_i|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,x=\mathbf{0}_n}) \\ &=\alpha \sum_i p_i(\!|t_i|\!)_{\theta,x=A} + \beta\sum_i p_i(\!|t_i|\!)_{\theta,x=B} - (\alpha+\beta-1)\sum_i p_i(\!|t_i|\!)_{\theta,x=\mathbf{0}_n} \\ &=\alpha(\!|\sum_i p_i t_i|\!)_{\theta,x=A} + \beta(\!|\sum_i p_i t_i|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|\sum_i p_i t_i|\!)_{\theta,x=\mathbf{0}_n}\end{aligned}$$

- Let $t = \mathsf{letcase}^\circ\ y = r\ \mathsf{in}\ \{t_1,\dots,t_m\}$. Replacing t in Equation 1, we want to show that:

$$\begin{aligned}(\!|\mathsf{letcase}^\circ\ y=r\ \mathsf{in}\ \{t_1,\dots,t_m\}|\!)_{\theta,x=\alpha A+\beta B} =&\alpha(\!|\mathsf{letcase}^\circ\ y=r\ \mathsf{in}\ \{t_1,\dots,t_m\}|\!)_{\theta,x=A} \\ &+\beta(\!|\mathsf{letcase}^\circ\ y=r\ \mathsf{in}\ \{t_1,\dots,t_m\}|\!)_{\theta,x=B} \\ &-(\alpha+\beta-1)(\!|\mathsf{letcase}^\circ\ y=r\ \mathsf{in}\ \{t_1,\dots,t_m\}|\!)_{\theta,x=\mathbf{0}_n}\end{aligned} \tag{7}$$

Since the type system is affine, either $x \in \mathsf{FV}(r)$ or $x \in \mathsf{FV}(t_i)$ holds, for some i.

  - If $x \in \mathsf{FV}(r)$, let $\rho^i_{\alpha A+\beta B}$, ρ^i_A, ρ^i_B, ρ^i_0 for $i \in \{1,\dots,m\}$ such that:

$$(\!|r|\!)_{\theta,x=\alpha A+\beta B} = \bigoplus_{i=1}^m \rho^i_{\alpha A+\beta B} \quad (\!|r|\!)_{\theta,x=A} = \bigoplus_{i=1}^m \rho^i_A \quad (\!|r|\!)_{\theta,x=B} = \bigoplus_{i=1}^m \rho^i_B \quad (\!|r|\!)_{\theta,x=\mathbf{0}_n} = \bigoplus_{i=1}^m \rho^i_0$$

By the inductive hypothesis on r we have:

$$(\!|r|\!)_{\theta,x=\alpha A+\beta B} = \alpha(\!|r|\!)_{\theta,x=A} + \beta(\!|r|\!)_{\theta,x=B} - (\alpha+\beta-1)(\!|r|\!)_{\theta,x=\mathbf{0}_n}$$

Then:

$$\bigoplus_{i=1}^m \rho^i_{\alpha A+\beta B} = \alpha\bigoplus_{i=1}^m \rho^i_A + \beta\bigoplus_{i=1}^m \rho^i_B - (\alpha+\beta-1)\bigoplus_{i=1}^m \rho^i_0$$

Thus, for all i we have:

$$\rho^i_{\alpha A+\beta B} = \alpha\rho^i_A + \beta\rho^i_B - (\alpha+\beta-1)\rho^i_0 \tag{8}$$

Applying the trace to both sides of the equation:

$$\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) = \alpha\mathsf{tr}\left(\rho^i_A\right) + \beta\mathsf{tr}\left(\rho^i_B\right) - (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right) \tag{9}$$

In general for all ρ, by the inductive hypothesis on t_i (with $\alpha = \frac{1}{\mathsf{tr}(\rho)}$, $A = \rho$, $\beta = 0$, and n' the correct dimension) we have that:

$$(\!|t_i|\!)_{\theta,y=\frac{\rho}{\mathsf{tr}(\rho)}} = \frac{1}{\mathsf{tr}\,(\rho)}(\!|t_i|\!)_{\theta,y=\rho} - (\frac{1}{\mathsf{tr}\,(\rho)} + 0 - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

Therefore,

$$(\!|t_i|\!)_{\theta,y=\frac{\rho}{\mathsf{tr}(\rho)}} = \frac{1}{\mathsf{tr}\,(\rho)}(\!|t_i|\!)_{\theta,y=\rho} + (1-\frac{1}{\mathsf{tr}\,(\rho)})(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

Multiplying both sides by $\mathsf{tr}\,(\rho)$ we have:

$$\mathsf{tr}\,(\rho)\,(\!|t_i|\!)_{\theta,y=\frac{\rho}{\mathsf{tr}(\rho)}} = (\!|t_i|\!)_{\theta,y=\rho} + (\mathsf{tr}\,(\rho)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}} \tag{10}$$

From Equation 8 we have that:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = (\!|t_i|\!)_{\theta,y=\alpha\rho^i_A+\beta\rho^i_B-(\alpha+\beta-1)\rho^i_0}$$

By the inductive hypothesis, with $\alpha'=\alpha$, $A'=\rho^i_A$, $\beta'=1$, $B'=\beta\rho^i_B-(\alpha+\beta-1)\rho^i_0$, we have that:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + (\!|t_i|\!)_{\theta,y=\beta\rho^i_B-(\alpha+\beta-1)\rho^i_0} - \alpha(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

Using the inductive hypothesis on the second term with: $\alpha'=\beta$, $A'=\rho^i_B$, $\beta'=-(\alpha+\beta-1)$ y $B'=\rho^i_0$ we have:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + \beta(\!|t_i|\!)_{\theta,y=\rho^i_B} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0} \tag{11}$$

By definition, and since $x\in\mathsf{FV}(r)$ but $x\notin\mathsf{FV}(t_i)$ for all i, to show that Equation 7 holds we want to show that:

$$\begin{aligned}\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_{\alpha A+\beta B}}} =& \alpha\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i_A\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_A}}\\ &+\beta\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i_B\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_B}}\\ &-(\alpha+\beta-1)\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i_0\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_0}}\end{aligned}$$

Where:

$$\widetilde{\phi} = \begin{cases}\mathbf{0}_{n'} & \text{if } \mathsf{tr}\,(\phi)=0\\ \frac{\phi}{\mathsf{tr}(\phi)} & \text{otherwise}\end{cases} \tag{12}$$

We are going to show equality of the summation term by term, meaning that for all i we have:

$$\begin{aligned}\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_{\alpha A+\beta B}}} =& \alpha\mathsf{tr}\left(\rho^i_A\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_A}}\\ &+\beta\mathsf{tr}\left(\rho^i_B\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_B}}\\ &-(\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right)(\!|t_i|\!)_{\theta,y=\widetilde{\rho^i_0}}\end{aligned} \tag{13}$$

1. Case $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)\neq 0$, $\mathsf{tr}\left(\rho^i_A\right)\neq 0$, $\mathsf{tr}\left(\rho^i_B\right)\neq 0$, $\mathsf{tr}\left(\rho^i_0\right)\neq 0$
   Evaluating Equation 13 we want to show that:

$$\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_{\alpha A+\beta B}}{\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)}} = \alpha\mathsf{tr}\left(\rho^i_A\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_A}{\mathsf{tr}\left(\rho^i_A\right)}}$$

$$+ \beta \mathsf{tr}\left(\rho^i_B\right) (\!|t_i|\!)_{\theta, y=\frac{\rho^i_B}{\mathsf{tr}\left(\rho^i_B\right)}}$$
$$- (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right) (\!|t_i|\!)_{\theta, y=\frac{\rho^i_0}{\mathsf{tr}\left(\rho^i_0\right)}}$$

Using Equation 10 this is equivalent to showing that:

$$\begin{aligned}(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} + (\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}} =& \alpha((\!|t_i|\!)_{\theta,y=\rho^i_A} + (\mathsf{tr}\left(\rho^i_A\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\\ &+ \beta((\!|t_i|\!)_{\theta,y=\rho^i_B} + (\mathsf{tr}\left(\rho^i_B\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\\ &- (\alpha+\beta-1)((\!|t_i|\!)_{\theta,y=\rho^i_0} + (\mathsf{tr}\left(\rho^i_0\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\end{aligned}$$

Reordering the terms this is equivalent to:

$$\begin{aligned}(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} =& \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + \beta(\!|t_i|\!)_{\theta,y=\rho^i_B} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0} +\\ &(-\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) + \alpha\mathsf{tr}\left(\rho^i_A\right) + \beta\mathsf{tr}\left(\rho^i_B\right) - (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right))(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}\end{aligned}$$

Using Equation 9 the last term is null and so:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + \beta(\!|t_i|\!)_{\theta,y=\rho^i_B} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0}$$

We get Equation 11.

2. Cases $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) \neq 0$, $\mathsf{tr}\left(\rho^i_A\right) = 0$, $\mathsf{tr}\left(\rho^i_B\right) \neq 0$, $\mathsf{tr}\left(\rho^i_0\right) \neq 0$ y $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) \neq 0$, $\mathsf{tr}\left(\rho^i_A\right) \neq 0$, $\mathsf{tr}\left(\rho^i_B\right) = 0$, $\mathsf{tr}\left(\rho^i_0\right) \neq 0$ (analogues)
   We prove the case for $\mathsf{tr}\left(\rho^i_B\right) = 0$. Evaluating in Equation 13, we want to show that:

$$\begin{aligned}\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) (\!|t_i|\!)_{\theta, y=\frac{\rho^i_{\alpha A+\beta B}}{\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)}} =& \alpha\mathsf{tr}\left(\rho^i_A\right) (\!|t_i|\!)_{\theta, y=\frac{\rho^i_A}{\mathsf{tr}\left(\rho^i_A\right)}}\\ &- (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right) (\!|t_i|\!)_{\theta, y=\frac{\rho^i_0}{\mathsf{tr}\left(\rho^i_0\right)}}\end{aligned}$$

   Using Equation 10 this is equivalent to:

$$\begin{aligned}(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} + (\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}} =& \alpha((\!|t_i|\!)_{\theta,y=\rho^i_A} + (\mathsf{tr}\left(\rho^i_A\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\\ &- (\alpha+\beta-1)((\!|t_i|\!)_{\theta,y=\rho^i_0} + (\mathsf{tr}\left(\rho^i_0\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\end{aligned}$$

   Reordering, this is equivalent to:

$$\begin{aligned}(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} =& \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0} +\\ &(-\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) + \alpha\mathsf{tr}\left(\rho^i_A\right) - (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right) + \beta)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}\end{aligned}$$

   Using Equation 9 with $\mathsf{tr}\left(\rho^i_B\right) = 0$, we have:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0} + \beta(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

   This is equivalent to Equation 11 in this case, because $\mathsf{tr}\left(\rho^i_B\right) = 0$ implies that $\rho^i_B = \mathbf{0}_{n'}$ since ρ^i_B is a positive matrix.

3. Case $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) \neq 0$, $\mathsf{tr}\left(\rho^i_A\right) \neq 0$, $\mathsf{tr}\left(\rho^i_B\right) \neq 0$, $\mathsf{tr}\left(\rho^i_0\right) = 0$
   Evaluating in Equation 13, we want to show that:

$$\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) (\!|t_i|\!)_{\theta, y=\frac{\rho^i_{\alpha A+\beta B}}{\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)}} = \alpha\mathsf{tr}\left(\rho^i_A\right) (\!|t_i|\!)_{\theta, y=\frac{\rho^i_A}{\mathsf{tr}\left(\rho^i_A\right)}}$$

$$+\beta\mathsf{tr}\left(\rho^i_B\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_B}{\mathsf{tr}\left(\rho^i_B\right)}}$$

Using Equation 10 this is equivalent to:

$$\begin{aligned}(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}}+(\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}} =&\alpha((\!|t_i|\!)_{\theta,y=\rho^i_A}+(\mathsf{tr}\left(\rho^i_A\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\\&+\beta((\!|t_i|\!)_{\theta,y=\rho^i_B}+(\mathsf{tr}\left(\rho^i_B\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\end{aligned}$$

Reordering, this is equivalent to:

$$\begin{aligned}(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}}=&\alpha(\!|t_i|\!)_{\theta,y=\rho^i_A}+\beta(\!|t_i|\!)_{\theta,y=\rho^i_B}+\\&(-\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)+\alpha\mathsf{tr}\left(\rho^i_A\right)+\beta\mathsf{tr}\left(\rho^i_B\right)+1-\alpha-\beta)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}\end{aligned}$$

Using Equation 9 evaluated according to this case, where $\mathsf{tr}\left(\rho^i_0\right)=0$:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}}=\alpha(\!|t_i|\!)_{\theta,y=\rho^i_A}+\beta(\!|t_i|\!)_{\theta,y=\rho^i_B}-(\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

This is equivalent to Equation 11 in this case, because $\mathsf{tr}\left(\rho^i_0\right)=0$ implies that $\rho^i_0=\mathbf{0}_{n'}$ since ρ^i_0 is a positive matrix.

4. Case $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)\neq 0$, $\mathsf{tr}\left(\rho^i_A\right)=0$, $\mathsf{tr}\left(\rho^i_B\right)=0$, $\mathsf{tr}\left(\rho^i_0\right)\neq 0$
Evaluating in Equation 13, we want to show that:

$$\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_{\alpha A+\beta B}}{\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)}}=-(\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_0}{\mathsf{tr}\left(\rho^i_0\right)}}$$

Using Equation 10 this is equivalent to:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}}+(\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}=-(\alpha+\beta-1)((\!|t_i|\!)_{\theta,y=\rho^i_0}+(\mathsf{tr}\left(\rho^i_0\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})$$

Reordering the terms:

$$\begin{aligned}(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}}=&-(\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0}+\\&(-\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)-(\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right)+\alpha+\beta)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}\end{aligned}$$

Using Equation 9 evaluated for this case, where $\mathsf{tr}\left(\rho^i_A\right)=\mathsf{tr}\left(\rho^i_B\right)=0$:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}}=-(\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0}+(\alpha+\beta)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

This is equivalent to Equation 11 in this case, because $\mathsf{tr}\left(\rho^i_A\right)=\mathsf{tr}\left(\rho^i_B\right)=0$ implies that $\rho^i_A=\rho^i_B=\mathbf{0}_{n'}$ since ρ^i_A and ρ^i_B are positive matrices.

5. Cases $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)\neq 0$, $\mathsf{tr}\left(\rho^i_A\right)=0$, $\mathsf{tr}\left(\rho^i_B\right)\neq 0$, $\mathsf{tr}\left(\rho^i_0\right)=0$ y $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)\neq 0$, $\mathsf{tr}\left(\rho^i_A\right)\neq 0$, $\mathsf{tr}\left(\rho^i_B\right)=0$, $\mathsf{tr}\left(\rho^i_0\right)=0$ (analogues)
We will prove the case where $\mathsf{tr}\left(\rho^i_B\right)=0$. Evaluating in Equation 13, we want to show that:

$$\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_{\alpha A+\beta B}}{\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)}}=\alpha\mathsf{tr}\left(\rho^i_A\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_A}{\mathsf{tr}\left(\rho^i_A\right)}}$$

Using Equation 10 this is equivalent to:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}}+(\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}=\alpha((\!|t_i|\!)_{\theta,y=\rho^i_A}+(\mathsf{tr}\left(\rho^i_A\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})$$

Reordering the terms:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + (-\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) + \alpha\mathsf{tr}\left(\rho^i_A\right) + 1 - \alpha)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

Using Equation 9 evaluated for this case, with $\mathsf{tr}\left(\rho^i_B\right) = \mathsf{tr}\left(\rho^i_0\right) = 0$:

$$(\!|t_i|\!)_{\theta,y=\rho^i_{\alpha A+\beta B}} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + (1-\alpha)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

This is the same as Equation 11 in this case, because $\mathsf{tr}\left(\rho^i_B\right) = \mathsf{tr}\left(\rho^i_0\right) = 0$ implies that $\rho^i_B = \rho^i_0 = \mathbf{0}_{n'}$ since ρ^i_B and ρ^i_0 are positive matrices.

6. Case $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) = 0$, $\mathsf{tr}\left(\rho^i_A\right) \neq 0$, $\mathsf{tr}\left(\rho^i_B\right) \neq 0$, $\mathsf{tr}\left(\rho^i_0\right) \neq 0$
Evaluating in Equation 13, with n'' the correct dimension, we want to show that:

$$\mathbf{0}_{n''} = \alpha\mathsf{tr}\left(\rho^i_A\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_A}{\mathsf{tr}\left(\rho^i_A\right)}} + \beta\mathsf{tr}\left(\rho^i_B\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_B}{\mathsf{tr}\left(\rho^i_B\right)}} - (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_0}{\mathsf{tr}\left(\rho^i_0\right)}}$$

Using Equation 10 this is equivalent to:

$$\begin{aligned}\mathbf{0}_{n''} =& \alpha((\!|t_i|\!)_{\theta,y=\rho^i_A} + (\mathsf{tr}\left(\rho^i_A\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\\ &+ \beta((\!|t_i|\!)_{\theta,y=\rho^i_B} + (\mathsf{tr}\left(\rho^i_B\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\\ &- (\alpha+\beta-1)((\!|t_i|\!)_{\theta,y=\rho^i_0} + (\mathsf{tr}\left(\rho^i_0\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})\end{aligned}$$

Reordering, this is equivalent to:

$$\begin{aligned}\mathbf{0}_{n''} =& \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + \beta(\!|t_i|\!)_{\theta,y=\rho^i_B} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0} +\\ &(\alpha\mathsf{tr}\left(\rho^i_A\right) + \beta\mathsf{tr}\left(\rho^i_B\right) - (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}\end{aligned}$$

Using Equation 9 evaluated for this case, with $\mathsf{tr}\left(\rho_\alpha A + \beta B^i\right) = 0$:

$$\mathbf{0}_{n''} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} + \beta(\!|t_i|\!)_{\theta,y=\rho^i_B} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0} - (\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

This is equivalent to Equation 11 in this case, because $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) = 0$ implies that $\rho^i_{\alpha A+\beta B} = \mathbf{0}_{n'}$ since $\rho^i_{\alpha A+\beta B}$ is a positive matrix.

7. Cases $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) = 0$, $\mathsf{tr}\left(\rho^i_A\right) = 0$, $\mathsf{tr}\left(\rho^i_B\right) \neq 0$, $\mathsf{tr}\left(\rho^i_0\right) \neq 0$ y $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) = 0$, $\mathsf{tr}\left(\rho^i_A\right) \neq 0$, $\mathsf{tr}\left(\rho^i_B\right) = 0$, $\mathsf{tr}\left(\rho^i_0\right) \neq 0$ (analogues)
We will prove the case where $\mathsf{tr}\left(\rho^i_B\right) = 0$. Evaluating in Equation 13, we want to show that:

$$\mathbf{0}_{n''} = \alpha\mathsf{tr}\left(\rho^i_A\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_A}{\mathsf{tr}\left(\rho^i_A\right)}} - (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_0}{\mathsf{tr}\left(\rho^i_0\right)}}$$

Using Equation 10 this is equivalent to:

$$\mathbf{0}_{n''} = \alpha((\!|t_i|\!)_{\theta,y=\rho^i_A} + (\mathsf{tr}\left(\rho^i_A\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}) - (\alpha+\beta-1)((\!|t_i|\!)_{\theta,y=\rho^i_0} + (\mathsf{tr}\left(\rho^i_0\right) - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})$$

Reordering the terms:

$$\mathbf{0}_{n''} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0} + (\alpha\mathsf{tr}\left(\rho^i_A\right) - (\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right) + \beta - 1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

Using Equation 9 evaluated for this case, with $\mathsf{tr}\left(\rho_\alpha A + \beta B^i\right) = \mathsf{tr}\left(\rho^i_B\right) = 0$:

$$\mathbf{0}_{n''} = \alpha(\!|t_i|\!)_{\theta,y=\rho^i_A} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta,y=\rho^i_0} + (\beta-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

This is equivalent to Equation 11 in this case, because $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right) = \mathsf{tr}\left(\rho^i_B\right) = 0$ implies that $\rho^i_{\alpha A+\beta B} = \rho^i_B = \mathbf{0}_{n'}$ since $\rho^i_{\alpha A+\beta B}$ and ρ^i_B are positive matrices.

8. Case $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)=0$, $\mathsf{tr}\left(\rho^i_A\right)\neq 0$, $\mathsf{tr}\left(\rho^i_B\right)\neq 0$, $\mathsf{tr}\left(\rho^i_0\right)=0$
Evaluating in Equation 13, we want to show that:

$$\mathbf{0}_{n''}=\alpha\mathsf{tr}\left(\rho^i_A\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_A}{\mathsf{tr}\left(\rho^i_A\right)}}+\beta\mathsf{tr}\left(\rho^i_B\right)(\!|t_i|\!)_{\theta,y=\frac{\rho^i_B}{\mathsf{tr}\left(\rho^i_B\right)}}$$

Using Equation 10 this is equivalent to:

$$\mathbf{0}_{n''}=\alpha((\!|t_i|\!)_{\theta,y=\rho^i_A}+(\mathsf{tr}\left(\rho^i_A\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})+\beta((\!|t_i|\!)_{\theta,y=\rho^i_B}+(\mathsf{tr}\left(\rho^i_B\right)-1)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}})$$

Reordering the terms:

$$\mathbf{0}_{n''}=\alpha(\!|t_i|\!)_{\theta,y=\rho^i_A}+\beta(\!|t_i|\!)_{\theta,y=\rho^i_B}+(\alpha\mathsf{tr}\left(\rho^i_A\right)+\beta\mathsf{tr}\left(\rho^i_B\right)-\alpha-\beta)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

Using Equation 9 evaluated for this case, with $\mathsf{tr}\left(\rho_\alpha A+\beta B^i\right)=\mathsf{tr}\left(\rho^i_0\right)=0$:

$$\mathbf{0}_{n''}=\alpha(\!|t_i|\!)_{\theta,y=\rho^i_A}+\beta(\!|t_i|\!)_{\theta,y=\rho^i_B}-(\alpha+\beta)(\!|t_i|\!)_{\theta,y=\mathbf{0}_{n'}}$$

This is equivalent to Equation 11 in this case, because $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)=\mathsf{tr}\left(\rho^i_0\right)=0$ implies that $\rho^i_{\alpha A+\beta B}=\rho^i_0=\mathbf{0}_{n'}$ since $\rho^i_{\alpha A+\beta B}$ and ρ^i_0 are positive matrices.

9. Trivial cases:
The case for $\mathsf{tr}\left(\rho^i_{\alpha A+\beta B}\right)=\mathsf{tr}\left(\rho^i_A\right)=\mathsf{tr}\left(\rho^i_B\right)=\mathsf{tr}\left(\rho^i_0\right)=0$ holds trivially.
The remaining cases are the ones such that $\mathsf{tr}\left(\rho^i_M\right)\neq 0$ for some $M\in\{\alpha A+\beta B, A, B, 0\}$ and the other traces are 0. But using Equation 9, we have:
   * (Case $M=\alpha A+\beta B$) $\mathsf{tr}\left(\rho_{\alpha A+\beta B}\right)=0$ and then Equation 13 holds.
   * (Case $M=A$) $\alpha\mathsf{tr}\left(\rho^i_A\right)=0$ and either $\alpha=0$ or $\mathsf{tr}\left(\rho^i_A\right)=0$. In both cases Equation 13 holds.
   * (Case $M=B$) $\beta\mathsf{tr}\left(\rho^i_B\right)=0$ and either $\beta=0$ or $\mathsf{tr}\left(\rho^i_B\right)=0$ and in both cases Equation 13 holds.
   * (Case $M=0$) $(\alpha+\beta-1)\mathsf{tr}\left(\rho^i_0\right)=0$ and either $(\alpha+\beta-1)=0$ or $\mathsf{tr}\left(\rho^i_0\right)=0$. In both cases Equation 13 holds.

- If $x\in\mathsf{FV}(t_i)$ for one or several i, let ρ^i such that:

$$(\!|r|\!)_\theta=\bigoplus_{i=1}^{n}\rho^i$$

From Equation 7, in this case we want to show that:

$$\begin{aligned}\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i\right)(\!|t_i|\!)_{\theta,x=\alpha A+\beta B,y=\widetilde{\rho^i}}=&\alpha\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i\right)(\!|t_i|\!)_{\theta,x=A,y=\widetilde{\rho^i}}\\&+\beta\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i\right)(\!|t_i|\!)_{\theta,x=B,y=\widetilde{\rho^i}}\\&-(\alpha+\beta-1)\sum_{i=1}^{n}\mathsf{tr}\left(\rho^i\right)(\!|t_i|\!)_{\theta,x=\mathbf{0}_n,y=\widetilde{\rho^i}}\end{aligned}$$

Let $\widetilde{\cdot}$ as defined in Equation 12, we will show that for all i the following holds:

$$\begin{aligned}\mathsf{tr}\left(\rho^i\right)(\!|t_i|\!)_{\theta,x=\alpha A+\beta B,y=\widetilde{\rho^i}}=&\alpha\mathsf{tr}\left(\rho^i\right)(\!|t_i|\!)_{\theta,x=A,y=\widetilde{\rho^i}}+\beta\mathsf{tr}\left(\rho^i\right)(\!|t_i|\!)_{\theta,x=B,y=\widetilde{\rho^i}}\\&-(\alpha+\beta-1)\mathsf{tr}\left(\rho^i\right)(\!|t_i|\!)_{\theta,x=\mathbf{0}_n,y=\widetilde{\rho^i}}\end{aligned}$$

If $\mathsf{tr}\left(\rho^i\right) = 0$, it holds trivially. Otherwise it can be proven that for all i:

$$\begin{aligned}(\!|t_i|\!)_{\theta,x=\alpha A+\beta B,y=\frac{\rho^i}{\mathsf{tr}(\rho^i)}} =&\alpha(\!|t_i|\!)_{\theta,x=A,y=\frac{\rho^i}{\mathsf{tr}(\rho^i)}} + \beta(\!|t_i|\!)_{\theta,x=B,y=\frac{\rho^i}{\mathsf{tr}(\rho^i)}} \\ &- (\alpha+\beta-1)(\!|t_i|\!)_{\theta,x=\mathbf{0}_n,y=\frac{\rho^i}{\mathsf{tr}(\rho^i)}}\end{aligned}$$

Let $\theta' = \theta \cup \{y = \frac{\rho^i}{\mathsf{tr}(\rho^i)}\}$, then the previous equation is the same as:

$$(\!|t_i|\!)_{\theta',x=\alpha A+\beta B} = \alpha(\!|t_i|\!)_{\theta',x=A} + \beta(\!|t_i|\!)_{\theta',x=B} - (\alpha+\beta-1)(\!|t_i|\!)_{\theta',x=\mathbf{0}_n}$$

This holds by inductive hypothesis on t_i. ◀

B.2 Proof of Lemma 4.2

▶ **Lemma 4.2** (Application). *If* $\Gamma, x : A \vdash t : B$ *and* $\theta \vDash \Gamma$, *then for all* $a \in \mathbb{C}^{\dim(A)\times\dim(A)}$ *we have* $(\!|\lambda x.t|\!)_\theta \;\#\; a = (\!|t|\!)_{\theta,x=a}$.

Proof. By definition we have:

$$\begin{aligned}(\!|\lambda x.t|\!)_\theta &= \overline{\chi}_{[a\mapsto(\!|t|\!)_{\theta,x=a}]} \\ &= \begin{pmatrix} (\!|t|\!)_{\theta,x=E^A_{11}} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} & \cdots & (\!|t|\!)_{\theta,x=E^A_{1n}} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} \\ \vdots & \ddots & \vdots \\ (\!|t|\!)_{\theta,x=E^A_{n1}} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} & \cdots & (\!|t|\!)_{\theta,x=E^A_{nn}} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} \end{pmatrix} \oplus (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}\end{aligned}$$

Let $\{E^A_{ij}\}$ be the canonical basis for $\mathbb{C}^{\dim(A)\times\dim(A)}$, decomposing a on this basis and applying $(\!|\lambda x.t|\!)_\theta$ through $\#$:

$$\begin{aligned}(\!|\lambda x.t|\!)_\theta \;\#\; a =&(\!|\lambda x.t|\!)_\theta \;\#\; \left(\sum_{i=1}^n\sum_{j=1}^n a_{ij}E^A_{ij}\right) \\ =&\sum_{i=1}^n\sum_{j=1}^n a_{ij}\left((\!|t|\!)_{\theta,x=E^A_{ij}} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}\right) + (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}\end{aligned}$$

By Lemma 4.1, $(\!|t|\!)_{\theta,x=E^A_{ij}} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}$ is linear on E^A_{ij}, then we have:

$$(\!|\lambda x.t|\!)_\theta \;\#\; a = (\!|t|\!)_{\theta,x=\sum_{ij}a_{ij}E^A_{ij}} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} + (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} = (\!|t|\!)_{\theta,x=a}$$ ◀

B.3 Proof of Lemma 4.3

▶ **Lemma 4.3** (Substitution). $(\!|t[x := r]|\!)_\theta = (\!|t|\!)_{\theta,x=(\!|r|\!)_\theta}$.

Proof. By induction on t.

- Let $t = x$, in this case we want to show that $(\!|x[x := r]|\!)_\theta = (\!|x|\!)_{\theta,x=(\!|r|\!)_\theta}$. On the one hand we have that $(\!|x[x := r]|\!)_\theta = (\!|r|\!)_\theta$ and on the other, $(\!|x|\!)_{\theta,x=(\!|r|\!)_\theta} = \theta'(x) = (\!|r|\!)_\theta$, where $\theta' = \theta \cup \{x = (\!|r|\!)_\theta\}$.
- Let $t = y \neq x$, we want to show that $(\!|y[x := r]|\!)_\theta = (\!|y|\!)_{\theta,x=(\!|r|\!)_\theta}$. Since $x \neq y$, we have that both sides are equal to $(\!|y|\!)_\theta$.

- Let $t = \lambda y.s$, in this case we want to show that $(\!|(\lambda y.s)[x := r]|\!)_\theta = (\!|\lambda y.s|\!)_{\theta,x=(\!|r|\!)_\theta}$. Rewriting the left-hand side:

$$(\!|(\lambda y.s)[x := r]|\!)_\theta = (\!|\lambda y.(s[x := r])|\!)_\theta = \overline{\chi}_{[a\mapsto(\!|s[x:=r]|\!)_{\theta,y=a}]}$$

  By the inductive hypothesis this is equal to $\overline{\chi}_{[a\mapsto(\!|s|\!)_{\theta,y=a,x=(\!|r|\!)_{\theta,y=a}}]}$, and since y is not a free variable in r, we have that $(\!|r|\!)_{\theta,y=a} = (\!|r|\!)_\theta$. Then this is equal to $\overline{\chi}_{[a\mapsto(\!|s|\!)_{\theta,y=a,x=(\!|r|\!)_\theta}]}$, that is the same as $(\!|\lambda y.s|\!)_{\theta,x=(\!|r|\!)_\theta}$.
- Let $t = s_1 s_2$, in this case we want to show that $(\!|(s_1 s_2)[x := r]|\!)_\theta = (\!|s_1 s_2|\!)_{\theta,x=(\!|r|\!)_\theta}$. Since the typing system is affine, either $x \in \mathsf{FV}(s_1)$ and $x \notin \mathsf{FV}(s_2)$, or $x \notin \mathsf{FV}(s_1)$ and $x \in \mathsf{FV}(s_2)$.
  - In the first case, rewriting the left-hand side:

$$(\!|(s_1 s_2)[x := r]|\!)_\theta = (\!|(s_1[x := r])s_2|\!)_\theta = (\!|s_1[x := r]|\!)_\theta \# (\!|s_2|\!)_\theta$$

    By the inductive hypothesis, this is equal to $(\!|s_1|\!)_{\theta,x=(\!|r|\!)_\theta} \# (\!|s_2|\!)_\theta$. Since $x \notin \mathsf{FV}(s_2)$, this is the same as $(\!|s_1|\!)_{\theta,x=(\!|r|\!)_\theta} \# (\!|s_2|\!)_{\theta,x=(\!|r|\!)_\theta} = (\!|s_1 s_2|\!)_{\theta,x=(\!|r|\!)_\theta}$.
  - In the second case, rewriting the left-hand side:

$$(\!|(s_1 s_2)[x := r]|\!)_\theta = (\!|s_1(s_2[x := r])|\!)_\theta = (\!|s_1|\!)_\theta \# (\!|s_2[x := r]|\!)_\theta$$

    By the inductive hypothesis, this is equal to $(\!|s_1|\!)_\theta \# (\!|s_2|\!)_{\theta,x=(\!|r|\!)_\theta}$. Since $x \notin \mathsf{FV}(s_1)$, this is the same as $(\!|s_1|\!)_{\theta,x=(\!|r|\!)_\theta} \# (\!|s_2|\!)_{\theta,x=(\!|r|\!)_\theta} = (\!|s_1 s_2|\!)_{\theta,x=(\!|r|\!)_\theta}$.
- Let $t = \mu_n y.s$. In this case, we want to show that $(\!|(\mu_n y.s)[x := r]|\!)_\theta = (\!|\mu_n y.s|\!)_{\theta,x=(\!|r|\!)_\theta}$. Rewriting the left-hand side we have:

$$(\!|(\mu_n y.s)[x := r]|\!)_\theta = (\!|\mu_n y.(s[x := r])|\!)_\theta = (\!|\lambda y.(s[x := r])|\!)_\theta \#_n \mathbf{0}_{\dim(A)}$$

  Using the previous abstraction case, this is equal to:

$$(\!|\lambda y.s|\!)_{\theta,x=(\!|r|\!)_\theta} \#_n \mathbf{0}_{\dim(A)} = (\!|\mu_n y.s|\!)_{\theta,x=(\!|r|\!)_\theta}$$

- Let $t = \perp$, this case is analogous to $t = y \neq x$.
- Let $t = \rho^n$, this case is analogous to $t = y \neq x$.
- Let $t = U^m s$, in this case we want to show that $(\!|(U^m s)[x := r]|\!)_\theta = (\!|U^m s|\!)_{\theta,x=(\!|r|\!)_\theta}$. Rewriting the left-hand side we have:

$$(\!|(U^m s)[x := r]|\!)_\theta = (\!|U^m(s[x := r])|\!)_\theta = \overline{U}(\!|s[x := r]|\!)_\theta \overline{U}^\dagger$$

  By the inductive hypothesis, this is equal to $\overline{U}(\!|s|\!)_{\theta,x=(\!|r|\!)_\theta}\overline{U}^\dagger = (\!|U^m s|\!)_{\theta,x=(\!|r|\!)_\theta}$.
- Let $t = \pi^m s$, in this case we want to show that $(\!|(\pi^m s)[x := r]|\!)_\theta = (\!|\pi^m s|\!)_{\theta,x=(\!|r|\!)_\theta}$. Rewriting the left-hand side we have:

$$(\!|(\pi^m s)[x := r]|\!)_\theta = (\!|\pi^m(s[x := r])|\!)_\theta = \bigoplus_{i=0}^{2^m-1}(\overline{|i\rangle\langle i|}(\!|s[x := r]|\!)_\theta \overline{|i\rangle\langle i|}^\dagger)$$

  By the inductive hypothesis, this is equal to $\bigoplus_{i=0}^{2^m-1}(\overline{|i\rangle\langle i|}(\!|s|\!)_{\theta,x=(\!|r|\!)_\theta}\overline{|i\rangle\langle i|}^\dagger) = (\!|\pi^m s|\!)_{\theta,x=(\!|r|\!)_\theta}$.
- Let $t = s_1 \otimes s_2$, in this case we want to show that $(\!|(s_1 \otimes s_2)[x := r]|\!)_\theta = (\!|s_1 \otimes s_2|\!)_{\theta,x=(\!|r|\!)_\theta}$. Since the typing system is affine, either $x \in \mathsf{FV}(s_1)$ and $x \notin \mathsf{FV}(s_2)$, or $x \notin \mathsf{FV}(s_1)$ and $x \in \mathsf{FV}(s_2)$.

- In the first case, rewriting the left-hand side we have:

$$(\!|(s_1 \otimes s_2)[x := r]|\!)_\theta = (\!|(s_1[x := r]) \otimes s_2|\!)_\theta = (\!|s_1[x := r]|\!)_\theta \otimes (\!|s_2|\!)_\theta$$

By the inductive hypothesis, this is equal to $(\!|s_1|\!)_{\theta,x=(\!|r|\!)_\theta} \otimes (\!|s_2|\!)_\theta$. Since $x \notin \mathsf{FV}(s_2)$ it holds that $(\!|s_2|\!)_\theta = (\!|s_2|\!)_{\theta,x=(\!|r|\!)_\theta}$, then:

$$(\!|(s_1 \otimes s_2)[x := r]|\!)_\theta = (\!|s_1|\!)_{\theta,x=(\!|r|\!)_\theta} \otimes (\!|s_2|\!)_{\theta,x=(\!|r|\!)_\theta} = (\!|s_1 \otimes s_2|\!)_{\theta,x=(\!|r|\!)_\theta}$$

- In the second case, rewriting the left-hand side we have:

$$(\!|(s_1 \otimes s_2)[x := r]|\!)_\theta = (\!|s_1 \otimes (s_2[x := r])|\!)_\theta = (\!|s_1|\!)_\theta \otimes (\!|s_2[x := r]|\!)_\theta$$

By the inductive hypothesis, this is equal to $(\!|s_1|\!)_\theta \otimes (\!|s_2|\!)_{\theta,x=(\!|r|\!)_\theta}$. Since $x \notin \mathsf{FV}(s_1)$ it holds that $(\!|s_1|\!)_\theta = (\!|s_1|\!)_{\theta,x=(\!|r|\!)_\theta}$, then:

$$(\!|(s_1 \otimes s_2)[x := r]|\!)_\theta = (\!|s_1|\!)_{\theta,x=(\!|r|\!)_\theta} \otimes (\!|s_2|\!)_{\theta,x=(\!|r|\!)_\theta} = (\!|s_1 \otimes s_2|\!)_{\theta,x=(\!|r|\!)_\theta}$$

- Let $t = \mathsf{letcase}^\circ\ y = s\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}$. In this case we want to show that $(\!|(\mathsf{letcase}^\circ\ y = s\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\})[x := r]|\!)_\theta = (\!|\mathsf{letcase}^\circ\ y = s\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}|\!)_{\theta,x=(\!|r|\!)_\theta}$.
Rewriting the right-hand side we have:

$$(\!|\mathsf{letcase}^\circ\ y = s\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}|\!)_{\theta,x=(\!|r|\!)_\theta} = \sum_{i=0}^{2^m-1} \mathsf{tr}\,(\rho_i)\ (\!|t_i|\!)_{\theta,x=(\!|r|\!)_\theta,y=\rho'_i} \tag{14}$$

where

$$(\!|s|\!)_{\theta,x=(\!|r|\!)_\theta} = \bigoplus_{i=0}^{2^m-1} \rho_i \qquad\qquad \rho'_i = \begin{cases} \frac{\rho_i}{\mathsf{tr}(\rho_i)} & \text{if } \mathsf{tr}\,(\rho_i) \neq 0 \\ \rho_i & \text{if } \mathsf{tr}\,(\rho_i) = 0 \end{cases} \tag{15}$$

Since the typing system is affine, either $x \in \mathsf{FV}(s)$ and $x \notin \mathsf{FV}(t_i)$ for all i, or $x \notin \mathsf{FV}(s)$ and there exists at least one j such that $x \in \mathsf{FV}(t_j)$.

  - In the first case, rewriting the left-hand side we have:

$$\begin{aligned}&(\!|(\mathsf{letcase}^\circ\ y = s\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\})[x := r]|\!)_\theta = \\ &\quad (\!|\mathsf{letcase}^\circ\ y = s[x := r]\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}|\!)_\theta = \sum_{i=0}^{2^m-1} \mathsf{tr}\,(\sigma_i)\ (\!|t_i|\!)_{\theta,y=\sigma'_i}\end{aligned}$$

  where $(\!|s[x := r]|\!)_\theta = \bigoplus_{i=0}^{2^m-1} \sigma_i$, $\sigma'_i = \frac{\sigma_i}{\mathsf{tr}(\sigma_i)}$ if $\mathsf{tr}\,(\sigma_i) \neq 0$ and $\sigma'_i = \sigma_i$ if $\mathsf{tr}\,(\sigma_i) = 0$. By the inductive hypothesis, we have that $(\!|s[x := r]|\!)_\theta = (\!|s|\!)_{\theta,x=(\!|r|\!)_\theta}$. Using Equation 15, this implies that $\sigma_i = \rho_i$ and $\sigma'_i = \rho'_i$ for all i.
  Therefore, and as $x \notin \mathsf{FV}(t_i)$ for all i:

$$\begin{aligned}\sum_{i=0}^{2^m-1} \mathsf{tr}\,(\sigma_i)\ (\!|t_i|\!)_{\theta,y=\sigma'_i} &= \sum_{i=0}^{2^m-1} \mathsf{tr}\,(\rho_i)\ (\!|t_i|\!)_{\theta,y=\rho'_i} \\ &= \sum_{i=0}^{2^m-1} \mathsf{tr}\,(\rho_i)\ (\!|t_i|\!)_{\theta,x=(\!|r|\!)_\theta,y=\rho'_i}\end{aligned}$$

  and we obtain equality with the right-hand side of Equation 14.

- In the second case, rewriting the left-hand side we have:

$$(\!|(\mathsf{letcase}^\circ\ y = s\ \mathsf{in}\ \{t_0, \ldots, t_{2^m-1}\})[x := r]|\!)_\theta =$$

$$(\!|\mathsf{letcase}^\circ\ y = s\ \mathsf{in}\ \{t_0[x := r], \ldots, t_{2^m-1}[x := r]\}|\!)_\theta = \sum_{i=0}^{2^m-1} \mathsf{tr}(\tau_i)\, (\!|t_i[x := r]|\!)_{\theta, y=\tau_i'}$$

where $(\!|s|\!)_\theta = \bigoplus_{i=0}^{2^m-1} \tau_i$, $\tau_i' = \frac{\tau_i}{\mathsf{tr}(\tau_i)}$ if $\mathsf{tr}(\tau_i) \neq 0$ and $\tau_i' = \tau_i$ if $\mathsf{tr}(\tau_i) = 0$. Since $x \notin \mathsf{FV}(s)$, it holds that $(\!|s|\!)_\theta = (\!|s|\!)_{\theta, x=(\!|r|\!)_\theta}$. Using Equation 15 this implies that $\tau_i = \rho_i$ and $\tau_i' = \rho_i'$ for all i.
Therefore we have that:

$$(\!|(\mathsf{letcase}^\circ\ y = s\ \mathsf{in}\ \{t_0, \ldots, t_{2^m-1}\})[x := r]|\!)_\theta = \sum_{i=0}^{2^m-1} \mathsf{tr}(\rho_i)\, (\!|t_i[x := r]|\!)_{\theta, y=\rho_i'}$$

By the inductive hypothesis on every t_i:

$$(\!|(\mathsf{letcase}^\circ\ y = s\ \mathsf{in}\ \{t_0, \ldots, t_{2^m-1}\})[x := r]|\!)_\theta = \sum_{i=0}^{2^m-1} \mathsf{tr}(\rho_i)\, (\!|t_i|\!)_{\theta, x=(\!|r|\!)_{\theta, y=\rho_i'}, y=\rho_i'}$$

And since $y \notin \mathsf{FV}(r)$, we obtain equality with the right-hand side of Equation 14.

- Let $t = \sum_i p_i t_i$, we want to show that $(\!|(\sum_i p_i t_i)[x := r]|\!)_\theta = (\!|\sum_i p_i t_i|\!)_{\theta, x=(\!|r|\!)_\theta}$:
Rewriting the left-hand side we have:

$$(\!|(\sum_i p_i t_i)[x := r]|\!)_\theta = (\!|\sum_i p_i (t_i[x := r])|\!)_\theta = \sum_i p_i (\!|t_i[x := r]|\!)_\theta$$

By the inductive hypothesis, this is equal to $\sum_i p_i (\!|t_i|\!)_{\theta, x=(\!|r|\!)_\theta} = (\!|\sum_i p_i t_i|\!)_{\theta, x=(\!|r|\!)_\theta}$. ◀

B.4 Proof of Lemma 4.4

We first need an auxiliary lemma. Let θ be a valuation and Γ a typing context. $\theta \vDash_{\dim} \Gamma$ if and only if for every pair $(x, A) \in \Gamma$, we have $\dim(\theta(x)) = \dim(A)$.

▶ **Lemma B.1.** *Let* $\Gamma \vdash t : A$ *and* $\theta \vDash_{\dim} \Gamma$, *then* $\dim((\!|t|\!)_\theta) = \dim(A)$.

Proof. By induction on t.

- Let $t = x$. Then $(x, A) \in \Gamma$ and as $\theta \vDash_{\dim} \Gamma$ we have that $\dim((\!|x|\!)_\theta) = \dim(\theta(x)) = \dim(A)$.
- Let $t = \lambda x.u$. In this case $A = B \multimap C$. By definition we have $(\!|\lambda x.u|\!)_\theta = \overline{\chi}_{[a \mapsto (\!|u|\!)_{\theta, x=a}]}$. Let $\{E_{ij}^B\}$ be the canonical basis of $\mathbb{C}^{\dim(B) \times \dim(B)}$. By inversion we have $\Gamma, x : B \vdash u : C$, and as $\theta \vDash_{\dim} \Gamma$, $\dim(E_{ij}^B) = \dim(B)$ and $\dim(\mathbf{0}_{\dim(B)}) = \dim(B)$ we have:

$$\theta \cup \{x = E_{ij}^B\} \vDash_{\dim} \Gamma, x : B$$
$$\theta \cup \{x = \mathbf{0}_{\dim(B)}\} \vDash_{\dim} \Gamma, x : B$$

Then by the inductive hypothesis:

$$\dim((\!|u|\!)_{\theta, x=E_{ij}^B}) = \dim(C)$$
$$\dim((\!|u|\!)_{\theta, x=\mathbf{0}_{\dim(B)}}) = \dim(C)$$

Finally, by definition of $\overline{\chi}_{[a \mapsto (\!|u|\!)_{\theta, x=a}]}$, we have that $\dim((\!|\lambda x.u|\!)_\theta) = \dim(B)\dim(C) + \dim(C) = \dim(B \multimap C)$.

- Let $t = uv$. Let $\Gamma_1, \Gamma_2 = \Gamma$ such that $\Gamma_1 \vdash u : B \multimap A$ and $\Gamma_2 \vdash v : B$; we have $\theta \vDash_{\text{dim}} \Gamma_1$ and $\theta \vDash_{\text{dim}} \Gamma_2$. By the inductive hypothesis, $\dim((\!|u|\!)_\theta) = \dim(B \multimap A)$ and $\dim((\!|v|\!)_\theta) = \dim(B)$. Since $\dim(B \multimap A) = (\dim(B) + 1)\dim(A)$, by definition of $\#$ we have that $\dim((\!|uv|\!)_\theta) = \dim((\!|u|\!)_\theta \# (\!|v|\!)_\theta) = \dim(A)$.
- Let $t = \mu_n x.u$. By inversion we have $\Gamma, x : A \vdash u : A$, and using rule $\multimap_i$ we have $\Gamma \vdash \lambda x.u : A \multimap A$. By definition we have $(\!|\mu_n x.u|\!)_\theta = (\!|\lambda x.u|\!)_\theta \#_n \mathbf{0}_{\dim(A)}$, and using the previous abstraction case we have $\dim((\!|\lambda x.u|\!)_\theta) = \dim(A \multimap A)$. Therefore, since $\dim(A \multimap A) = (\dim(A) + 1)\dim(A)$ and every $\#$ application rewrites to a matrix in $\mathbb{C}^{\dim(A)\times\dim(A)}$, we have that $\dim((\!|\mu_n x.u|\!)_\theta) = \dim(A)$.
- Let $t = \bot$. By definition $\dim((\!|\bot|\!)_\theta) = \dim(\mathbf{0}_{\dim(A)}) = \dim(A)$ holds.
- Let $t = \rho^n$. In this case $A = n$ and $\dim(\rho) = 2^n$.
- Let $t = U^m v$. In this case $A = n$. By inversion $\Gamma \vdash v : n$ and by the inductive hypothesis we have $\dim((\!|v|\!)_\theta) = 2^n$. Thus $\dim((\!|U^m v|\!)_\theta) = \dim(\overline{U}(\!|v|\!)_\theta\overline{U}^\dagger) = 2^n$.
- Let $t = \pi^m u$. In this case $A = (m, n)$. By inversion $\Gamma \vdash u : n$ and by the inductive hypothesis we have $\dim((\!|u|\!)_\theta) = 2^n$. Therefore $\dim((\!|\pi^m u|\!)_\theta) = \dim(\bigoplus_{i=0}^{2^m-1} \overline{|i\rangle\langle i|}(\!|u|\!)_\theta\overline{|i\rangle\langle i|}^\dagger) = 2^m 2^n$, since $\dim(\overline{|i\rangle\langle i|}(\!|u|\!)_\theta\overline{|i\rangle\langle i|}^\dagger) = 2^n$ for all i.
- Let $t = u \otimes v$. In this case $A = n + m$. Let $\Gamma_1, \Gamma_2 = \Gamma$ such that $\Gamma_1 \vdash u : n$ and $\Gamma_2 \vdash v : m$, by the inductive hypothesis since $\theta \vDash_{\text{dim}} \Gamma_1$ and $\theta \vDash_{\text{dim}} \Gamma_2$ we have $\dim((\!|u|\!)_\theta) = 2^n$ and $\dim((\!|v|\!)_\theta) = 2^m$. Therefore $\dim((\!|u \otimes v|\!)_\theta) = \dim((\!|u|\!)_\theta \otimes (\!|v|\!)_\theta) = 2^{n+m}$.
- Let $t = \sum_{i=1}^n p_i t_i$. By inversion $\Gamma \vdash t_i : A$ for all i, and by the inductive hypothesis $\dim((\!|t_i|\!)_\theta) = \dim(A)$. Therefore $\dim((\!|\sum_{i=1}^n p_i t_i|\!)_\theta) = \dim(\sum_{i=1}^n p_i (\!|t_i|\!)_\theta) = \dim(A)$.
- Let $t = \mathsf{letcase}^\circ\ x = r\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}$. Let $\Gamma_0, \dots, \Gamma_{2^m-1}, \Gamma' = \Gamma$ such that $\Gamma_i, x : n \vdash t_i : A$ for all i and $\Gamma' \vdash r : (m, n)$. Therefore $\theta \vDash_{\text{dim}} \Gamma'$ and by the inductive hypothesis $\dim((\!|r|\!)_\theta) = \dim((m, n)) = 2^{n+m}$. Let ρ_i with $0 \le i \le 2^m - 1$ be every one of the $2^n \times 2^n$ submatrices on $(\!|r|\!)_\theta$'s diagonal; we define ρ_i' as ρ_i if $\mathsf{tr}(\rho_i) = 0$, and as $\frac{\rho_i}{\mathsf{tr}(\rho_i)}$ otherwise. Since $\dim(\rho_i') = 2^n$ we have $\theta \cup \{x = \rho_i'\} \vDash_{\text{dim}} \Gamma_i, x : n$ for all i and by the inductive hypothesis $\dim(t_i) = \dim(A)$. Therefore $\dim((\!|\mathsf{letcase}^\circ\ x = r\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}|\!)_\theta) = \dim(\sum_{i=0}^{2^m-1} \mathsf{tr}(\rho_i)\,(\!|t_i|\!)_{\theta, x=\rho_i'}) = \dim(A)$. ◀

▶ **Lemma 4.4** (Reduction correctness). *If $\Gamma \vdash t : A$, $\theta \vDash \Gamma$, and $t \longrightarrow r$ then $(\!|t|\!)_\theta = (\!|r|\!)_\theta$.*

Proof. By induction on the relation $\longrightarrow$.

- $(\lambda x.t)r \longrightarrow t[x := r]$
  In this case we want to show that $(\!|(\lambda x.t)r|\!)_\theta = (\!|t[x := r]|\!)_\theta$. By definition we have that $(\!|(\lambda x.t)r|\!)_\theta = (\!|\lambda x.t|\!)_\theta \# (\!|r|\!)_\theta$. By Lemma 4.2 this is equal to $(\!|t|\!)_{\theta, x=(\!|r|\!)_\theta}$. By Lemma 4.3 it holds that $(\!|t|\!)_{\theta, x=(\!|r|\!)_\theta} = (\!|t[x := r]|\!)_\theta$.
- $\mu_0 x.t \longrightarrow \bot$
  In this case we want to show that $(\!|\mu_0 x.t|\!)_\theta = (\!|\bot|\!)_\theta$. By definition, we have $(\!|\mu_0 x.t|\!)_\theta = (\!|\lambda x.t|\!)_\theta \#_0 \mathbf{0}_{\dim(A)}$. This is equal to $\mathbf{0}_{\dim(A)} = (\!|\bot|\!)_\theta$.
- $\mu_{n+1} x.t \longrightarrow t[x := \mu_n x.t]$
  In this case we want to show that $(\!|\mu_{n+1} x.t|\!)_\theta = (\!|t[x := \mu_n x.t]|\!)_\theta$. By definition we have $(\!|\mu_{n+1} x.t|\!)_\theta = (\!|\lambda x.t|\!)_\theta \#_{n+1} \mathbf{0}_{\dim(A)}$. This is the same as:

  $$(\!|\lambda x.t|\!)_\theta \# ((\!|\lambda x.t|\!)_\theta \#_n \mathbf{0}_{\dim(A)})$$

  By the inductive hypothesis, this is equal to $(\!|\lambda x.t|\!)_\theta \# (\!|\mu_n x.t|\!)_\theta$. Since $\Gamma \vdash \mu_{n+1} x.t : A$, by inversion $\Gamma, x : A \vdash t : A$, and using rule μ we have $\Gamma \vdash \mu_n x.t : A$. Since $\theta \vDash \Gamma$ we also have $\theta \vDash_{\text{dim}} \Gamma$ and using Lemma B.1, $\dim((\!|\mu_n x.t|\!)_\theta) = \dim(A)$. Using Lemma 4.2 we have $(\!|\lambda x.t|\!)_\theta \# (\!|\mu_n x.t|\!)_\theta = (\!|t|\!)_{\theta, x=(\!|\mu_n x.t|\!)_\theta}$. Using Lemma 4.3, this is equal to $(\!|t[x := \mu_n x.t]|\!)_\theta$.

- $pt + q\bot \longrightarrow pt$
  Rewriting, $(\!|pt + q\bot|\!)_\theta = p(\!|t|\!)_\theta + q(\!|\bot|\!)_\theta = p(\!|t|\!)_\theta + q\mathbf{0}_{\dim(A)} = p(\!|t|\!)_\theta = (\!|pt|\!)_\theta$.
- $\bot\ t \longrightarrow \bot$
  Rewriting $(\!|\bot\ t|\!)_\theta$ we have:

$$(\!|\bot\ t|\!)_\theta = (\!|\bot|\!)_\theta \ \#\ (\!|t|\!)_\theta = \mathbf{0}_{\dim(A\multimap B)} \ \#\ (\!|t|\!)_\theta = \mathbf{0}_{\dim(B)} = (\!|\bot|\!)_\theta$$

- $U^m\rho^n \longrightarrow (\overline{U}\rho\overline{U}^\dagger)^n$
  By definition: $(\!|U^m\rho^n|\!)_\theta = \overline{U}(\!|\rho^n|\!)_\theta\overline{U}^\dagger = \overline{U}\rho\overline{U}^\dagger = (\!|(\overline{U}\rho\overline{U}^\dagger)^n|\!)_\theta$.
- $\rho^n \otimes \sigma^m \longrightarrow (\rho\otimes\sigma)^{n\times m}$
  By definition: $(\!|\rho^n \otimes \sigma^m|\!)_\theta = (\!|\rho^n|\!)_\theta \otimes (\!|\sigma^m|\!)_\theta = \rho\otimes\sigma = (\!|(\rho\otimes\sigma)^{n\times m}|\!)_\theta$.
- $\sum_i p_i\rho_i^n \longrightarrow (\sum_i p_i\rho_i)^n$
  By definition: $(\!|\sum_i p_i\rho_i^n|\!)_\theta = \sum_i p_i(\!|\rho_i^n|\!)_\theta = \sum_i p_i\rho_i = (\!|(\sum_i p_i\rho_i)^n|\!)_\theta$.
- $\sum_i (p_i t) \longrightarrow (\sum_i p_i)t$
  In this case we want to show that $(\!|\sum_i(p_i t)|\!)_\theta = (\!|(\sum_i p_i)t|\!)_\theta$. Both sides are equal to $\sum_i p_i(\!|t|\!)_\theta$, taking $\sum_i p_i$ as the only probability in the sum on the right-hand side.
- $(\sum_i p_i t_i)r \longrightarrow \sum_i p_i(t_i r)$
  In this case we want to show that $(\!|(\sum_i p_i t_i)r|\!)_\theta = (\!|\sum_i p_i(t_i r)|\!)_\theta$. Rewriting the left-hand side and using Lemma 3.7:

$$\begin{aligned}(\!|(\sum_i p_i t_i)r|\!)_\theta &= (\!|\sum_i p_i t_i|\!)_\theta \ \#\ (\!|r|\!)_\theta = (\sum_i p_i(\!|t_i|\!)_\theta) \ \#\ (\!|r|\!)_\theta \\ &\overset{(3.7)}{=} \sum_i p_i\left((\!|t_i|\!)_\theta \ \#\ (\!|r|\!)_\theta\right) = \sum_i p_i(\!|t_i r|\!)_\theta = (\!|\sum_i p_i(t_i r)|\!)_\theta\end{aligned}$$

- $\mathsf{letcase}^\circ\ x = \pi^m\rho^n\ \mathsf{in}\ \{t_0,\dots,t_{2^m-1}\} \longrightarrow \sum_{i=0}^{2^m-1} p_i t_i[x := \rho_i^n]$ where:

$$p_i = \mathsf{tr}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right) \qquad \rho_i = \begin{cases} \frac{\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger}{p_i} & \text{if } p_i \neq 0 \\ \overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger & \text{if } p_i = 0\end{cases}$$

  In this case we want to show that:

$$(\!|\mathsf{letcase}^\circ\ x = \pi^m\rho^n\ \mathsf{in}\ \{t_0,\dots,t_{2^m-1}\}|\!)_\theta = (\!|\sum_{i=0}^{2^m-1} p_i t_i[x := \rho_i^n]|\!)_\theta$$

  We have $(\!|\mathsf{letcase}^\circ\ x = \pi^m\rho^n\ \mathsf{in}\ \{t_0,\dots,t_{2^m-1}\}|\!)_\theta = \sum_{i=0}^{2^m-1} p_i(\!|t_i|\!)_{\theta,x=\rho_i}$ using the previous definitions for p_i and ρ_i, because $(\!|\pi^m\rho^n|\!)_\theta = \bigoplus_{i=0}^{2^m-1} \overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$.
  We also have that $(\!|\sum_{i=0}^{2^m-1} p_i t_i[x := \rho_i^n]|\!)_\theta = \sum_{i=0}^{2^m-1} p_i(\!|t_i[x := \rho_i^n]|\!)_\theta$ by definition. Using Substitution (Lemma 4.3) this is equal to $\sum_{i=0}^{2^m-1} p_i(\!|t_i|\!)_{\theta,x=(\!|\rho_i^n|\!)_\theta}$. Since $(\!|\rho_i^n|\!)_\theta = \rho_i$ for every valuation θ, this is equal to $\sum_{i=0}^{2^m-1} p_i(\!|t_i|\!)_{\theta,x=\rho_i}$.

Contextual cases:

- $t \longrightarrow r \implies ts \longrightarrow rs$
  In this case we want to show that $(\!|ts|\!)_\theta = (\!|rs|\!)_\theta$ when $t \longrightarrow r$.
  By definition, $(\!|ts|\!)_\theta = (\!|t|\!)_\theta \ \#\ (\!|s|\!)_\theta$. By the inductive hypothesis, since $t \longrightarrow r$, $(\!|t|\!)_\theta = (\!|r|\!)_\theta$ and so $(\!|t|\!)_\theta \ \#\ (\!|s|\!)_\theta = (\!|r|\!)_\theta \ \#\ (\!|s|\!)_\theta$. This is by definition the same as $(\!|rs|\!)_\theta$.
- $t \longrightarrow r \implies st \longrightarrow sr$
  In this case we want to show that $(\!|st|\!)_\theta = (\!|sr|\!)_\theta$ when $t \longrightarrow r$.
  By definition, $(\!|st|\!)_\theta = (\!|s|\!)_\theta \ \#\ (\!|t|\!)_\theta$. By the inductive hypothesis, since $t \longrightarrow r$, $(\!|t|\!)_\theta = (\!|r|\!)_\theta$ and so $(\!|s|\!)_\theta \ \#\ (\!|t|\!)_\theta = (\!|s|\!)_\theta \ \#\ (\!|r|\!)_\theta$. This is by definition the same as $(\!|sr|\!)_\theta$.

- $t \longrightarrow r \implies U^m t \longrightarrow U^m r$
  In this case we want to show that $(\!|U^m t|\!)_\theta = (\!|U^m r|\!)_\theta$ when $t \longrightarrow r$.
  By definition, $(\!|U^m t|\!)_\theta = \overline{U}(\!|t|\!)_\theta\overline{U}^\dagger$. By the inductive hypothesis, since $t \longrightarrow r$, $(\!|t|\!)_\theta = (\!|r|\!)_\theta$.
  Rewriting we have $\overline{U}(\!|r|\!)_\theta\overline{U}^\dagger$, and this is equal to $(\!|U^m r|\!)_\theta$ by definition.
- $t \longrightarrow r \implies \pi^m t \longrightarrow \pi^m r$
  In this case we want to show that $(\!|\pi^m t|\!)_\theta = (\!|\pi^m r|\!)_\theta$ when $t \longrightarrow r$.
  By definition, $(\!|\pi^m t|\!)_\theta = \bigoplus_{i=0}^{2^m-1}(\overline{|i\rangle\langle i|}(\!|t|\!)_\theta\overline{|i\rangle\langle i|}^\dagger)$. The inductive hypothesis in this case is $(\!|t|\!)_\theta = (\!|r|\!)_\theta$, then the term is the same as $\bigoplus_{i=0}^{2^m-1}(\overline{|i\rangle\langle i|}(\!|r|\!)_\theta\overline{|i\rangle\langle i|}^\dagger)$. By definition this is equal to $(\!|\pi^m r|\!)_\theta$.
- $t \longrightarrow r \implies t \otimes s \longrightarrow r \otimes s$
  In this case we want to show that $(\!|t \otimes s|\!)_\theta = (\!|r \otimes s|\!)_\theta$ when $t \longrightarrow r$.
  By definition, $(\!|t\otimes s|\!)_\theta = (\!|t|\!)_\theta \otimes (\!|s|\!)_\theta$. This is equal to $(\!|r|\!)_\theta \otimes (\!|s|\!)_\theta$ by the induction hypothesis, and by definition this is the same as $(\!|r \otimes s|\!)_\theta$.
- $t \longrightarrow r \implies s \otimes t \longrightarrow s \otimes r$
  In this case we want to show that $(\!|s \otimes t|\!)_\theta = (\!|s \otimes r|\!)_\theta$ when $t \longrightarrow r$.
  By definition, $(\!|s\otimes t|\!)_\theta = (\!|s|\!)_\theta \otimes (\!|t|\!)_\theta$. This is equal to $(\!|s|\!)_\theta \otimes (\!|r|\!)_\theta$ by the inductive hypothesis, and by definition this is the same as $(\!|s \otimes r|\!)_\theta$.
- $t_j \longrightarrow r_j$ for some j in $\{1,\dots,n\} \implies$
  $$\sum_{i=1}^n p_i t_i \longrightarrow \sum_{i=1}^n p_i r_i \text{ with } t_i = r_i \text{ for all } i \neq j \text{ in } \{1,\dots,n\}$$
  In this case we want to show that $(\!|\sum_{i=1}^n p_i t_i|\!)_\theta = (\!|\sum_{i=1}^n p_i r_i|\!)_\theta$ with $t_i = r_i$ for all $i \neq j$ in $\{1,\dots,n\}$ when $t_j \longrightarrow r_j$ for some j in $\{1,\dots,n\}$.
  By definition, $(\!|\sum_{i=1}^n p_i t_i|\!)_\theta = \sum_{i=1}^n p_i(\!|t_i|\!)_\theta$. This is the same as $\sum_{\substack{i=1\\i\neq j}}^n p_i(\!|t_i|\!)_\theta + p_j(\!|t_j|\!)_\theta$.
  Since $t_j \longrightarrow r_j$, by the inductive hypothesis $(\!|t_j|\!)_\theta = (\!|r_j|\!)_\theta$. Defining $r_i = t_i$ for all $i \neq j$, the sum term is equal to $\sum_{\substack{i=1\\i\neq j}}^n p_i(\!|r_i|\!)_\theta + p_j(\!|r_j|\!)_\theta = \sum_{i=1}^n p_i(\!|r_i|\!)_\theta$ and by definition this is the same as $(\!|\sum_{i=1}^n p_i r_i|\!)_\theta$.
- $t \longrightarrow r \implies \mathsf{letcase}^\circ\ x = t \text{ in } \{s_0,\dots,s_{2^m-1}\} \longrightarrow \mathsf{letcase}^\circ\ x = r \text{ in } \{s_0,\dots,s_{2^m-1}\}$
  We want to show that, when $t \longrightarrow r$, we have $(\!|\mathsf{letcase}^\circ\ x = t \text{ in } \{s_0,\dots,s_{2^m-1}\}|\!)_\theta = (\!|\mathsf{letcase}^\circ\ x = r \text{ in } \{s_0,\dots,s_{2^m-1}\}|\!)_\theta$.
  By definition, $(\!|\mathsf{letcase}^\circ\ x = t \text{ in } \{s_0,\dots,s_{2^m-1}\}|\!)_\theta = \sum_{i=0}^{2^m-1} \mathsf{tr}(\rho_i)\,(\!|s_i|\!)_{\theta,x=\rho'_i}$ where $(\!|t|\!)_\theta = \bigoplus_{i=0}^{2^m-1}\rho_i$, $\rho'_i = \frac{\rho_i}{\mathsf{tr}(\rho_i)}$ if $\mathsf{tr}(\rho_i) \neq 0$ and $\rho'_i = \rho_i$ if $\mathsf{tr}(\rho_i) = 0$.
  Also by definition, $(\!|\mathsf{letcase}^\circ\ x = r \text{ in } \{s_0,\dots,s_{2^m-1}\}|\!)_\theta = \sum_{i=0}^{2^m-1} \mathsf{tr}(\sigma_i)\,(\!|s_i|\!)_{\theta,x=\sigma'_i}$ where $(\!|r|\!)_\theta = \bigoplus_{i=0}^{2^m-1}\sigma_i$, $\sigma'_i = \frac{\sigma_i}{\mathsf{tr}(\sigma_i)}$ if $\mathsf{tr}(\sigma_i) \neq 0$ and $\sigma'_i = \sigma_i$ if $\mathsf{tr}(\sigma_i) = 0$.
  By the inductive hypothesis we have $(\!|t|\!)_\theta = (\!|r|\!)_\theta$, therefore $\rho_i = \sigma_i$ and $\rho'_i = \sigma'_i$ for all i.
  Then, $\sum_{i=0}^{2^m-1} \mathsf{tr}(\rho_i)\,(\!|s_i|\!)_{\theta,x=\rho'_i} = \sum_{i=0}^{2^m-1} \mathsf{tr}(\sigma_i)\,(\!|s_i|\!)_{\theta,x=\sigma'_i}$. ◀

B.5 Proof of Lemma 4.5

▶ **Lemma 4.5.** *Let $\Gamma, x : A \vdash t : B$, and for all $\theta \vDash \Gamma$ and $a \in (\!|A|\!)$, let $(\!|t|\!)_{\theta,x=a} \in (\!|B|\!)$. Then, the map $F^{t,x}_{\theta\vDash\Gamma} = a \mapsto (\!|t|\!)_{\theta,x=a} - (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}}$ is a CPM.*

Proof. We proceed by induction on t.

- Let $t = y \neq x$, then $\theta = \theta' \cup \{y = c\}$ and $F^{y,x}_{\theta\vDash\Gamma,y:C} = a \mapsto (c - c) = \mathbf{0}_n$, which is completely positive.
- Let $t = x$, then $F^{x,x}_{\theta\vDash\Gamma} = a \mapsto (a - \mathbf{0}_n) = I_n$, which is completely positive.
- Let $t = \lambda y.r$. Then, $B = C \multimap D$, and, by inversion , $\Gamma, x : A, y : C \vdash r : D$. Hence, by the induction hypothesis, we have that $F^{r,y}_{\tau\vDash\Gamma,x:A}$ and $F^{r,x}_{\delta\vDash\Gamma,y:C}$ are CPMs.

We need to prove that $F^{\lambda y.r,x}_{\theta\vDash\Gamma}$ is a CPM. This map is defined by

$$
\begin{aligned}
&F^{\lambda y.r,x}_{\theta\vDash\Gamma}(a)\\
&= \llparenthesis\lambda y.r\rrparenthesis_{\theta,x=a} - \llparenthesis\lambda y.r\rrparenthesis_{\theta,x=\mathbf{0}_n}\\
&= \overline{\chi}_{[c\mapsto\llparenthesis r\rrparenthesis_{\theta,x=a,y=c}]} - \overline{\chi}_{[c\mapsto\llparenthesis r\rrparenthesis_{\theta,x=\mathbf{0}_n,y=c}]}\\
&= \chi_{[c\mapsto\llparenthesis r\rrparenthesis_{\theta,x=a,y=c}-\llparenthesis r\rrparenthesis_{\theta,x=a,y=\mathbf{0}_m}]} \oplus \llparenthesis r\rrparenthesis_{\theta,x=a,y=\mathbf{0}_m} \quad (\text{with } m=\dim(C))\\
&- \chi_{[c\mapsto\llparenthesis r\rrparenthesis_{\theta,x=\mathbf{0}_n,y=c}-\llparenthesis r\rrparenthesis_{\theta,x=\mathbf{0}_n,y=\mathbf{0}_m}]} \oplus \llparenthesis r\rrparenthesis_{\theta,x=\mathbf{0}_n,y=\mathbf{0}_m}\\
&= \chi_{[c\mapsto(\llparenthesis r\rrparenthesis_{\theta,x=a,y=c}-\llparenthesis r\rrparenthesis_{\theta,x=a,y=\mathbf{0}_m})-(\llparenthesis r\rrparenthesis_{\theta,x=\mathbf{0}_n,y=c}-\llparenthesis r\rrparenthesis_{\theta,x=\mathbf{0}_n,y=\mathbf{0}_m})]} \oplus (\llparenthesis r\rrparenthesis_{\theta,x=a,y=\mathbf{0}_m} - \llparenthesis r\rrparenthesis_{\theta,x=\mathbf{0}_n,y=\mathbf{0}_m})\\
&= \chi_{[F^{r,y}_{\theta\cup\{x=a\}\vDash\Gamma,x:A}-F^{r,y}_{\theta\cup\{x=\mathbf{0}_n\}\vDash\Gamma,x:A}]} \oplus F^{r,x}_{\theta\cup\{y=\mathbf{0}_m\}\vDash\Gamma,y:C}(a)
\end{aligned}
$$

Let

$$
\begin{aligned}
&G(b) = F^{r,y}_{\theta\cup\{x=b\}\vDash\Gamma,x:A} = c\mapsto \llparenthesis r\rrparenthesis_{\theta,x=b,y=c} - \llparenthesis r\rrparenthesis_{\theta,x=b,y=\mathbf{0}_n}\\
&H(a) = G(a) - G(\mathbf{0}_n)
\end{aligned}
$$

Since G is a CPM, $H(a)$ is also a CPM. Then, $\chi_{[F^{r,y}_{\theta\cup\{x=a\}\vDash\Gamma,x:A}-F^{r,y}_{\theta\cup\{x=\mathbf{0}_n\}\vDash\Gamma,x:A}]} = \chi_{[H(a)]}$ is a CPM. Finally, the sum of CPMs is a CPM.

- Let $t = rs$. Then, by inversion, there are two cases:
  - $\Gamma, x : A \vdash r : C \multimap D$ and $\Delta \vdash s : C$. Hence, by the induction hypothesis, we have that $F^{r,x}_{\gamma\vDash\Gamma}$ is a CPM.
    We need to prove that $F^{rs,x}_{\theta\vDash\Gamma,\Delta}$, where $\theta = \gamma, \delta$ with $\gamma \vDash \Gamma$ and $\delta \vDash \Delta$, is a CPM.

$$
\begin{aligned}
F^{rs,x}_{\theta\vDash\Gamma,\Delta}(a) &= \llparenthesis rs\rrparenthesis_{\theta,x=a} - \llparenthesis rs\rrparenthesis_{\theta,x=\mathbf{0}_n}\\
&= \llparenthesis r\rrparenthesis_{\theta,x=a} \mathbin{\#} \llparenthesis s\rrparenthesis_{\theta,x=a} - \llparenthesis r\rrparenthesis_{\theta,x=\mathbf{0}_n} \mathbin{\#} \llparenthesis s\rrparenthesis_{\theta,x=\mathbf{0}_n}\\
&= \llparenthesis r\rrparenthesis_{\gamma,x=a} \mathbin{\#} \llparenthesis s\rrparenthesis_{\delta} - \llparenthesis r\rrparenthesis_{\gamma,x=\mathbf{0}_n} \mathbin{\#} \llparenthesis s\rrparenthesis_{\delta}\\
(\text{Lemma 3.7}) &= (\llparenthesis r\rrparenthesis_{\gamma,x=a} - \llparenthesis r\rrparenthesis_{\gamma,x=\mathbf{0}_n}) \mathbin{\#} \llparenthesis s\rrparenthesis_{\delta}\\
&= F^{r,x}_{\gamma\vDash\Gamma}(a) \mathbin{\#} \llparenthesis s\rrparenthesis_{\delta}
\end{aligned}
$$

    Notice that since $F^{r,x}_{\gamma\vDash\Gamma}(a)$ is a Choi matrix, then the application $\#$ is just the standard application, and since $\llparenthesis s\rrparenthesis_\delta \in \llparenthesis B\rrparenthesis$, it is positive, so $F^{r,x}_{\gamma\vDash\Gamma}(a) \mathbin{\#} \llparenthesis s\rrparenthesis_\delta$ is a CPM.
  - $\Gamma \vdash r : C \multimap D$ and $\Delta, x : A \vdash s : C$ Hence, by the induction hypothesis, we have that $F^{s,x}_{\delta\vDash\Delta}$ is a CPM.
    We need to prove that $F^{rs,x}_{\theta\vDash\Gamma,\Delta}$, where $\theta = \gamma, \delta$ with $\gamma \vDash \Gamma$ and $\delta \vDash \Delta$, is a CPM.

$$
\begin{aligned}
F^{rs,x}_{\theta\vDash\Gamma,\Delta}(a) &= \llparenthesis rs\rrparenthesis_{\theta,x=a} - \llparenthesis rs\rrparenthesis_{\theta,x=\mathbf{0}_n}\\
&= \llparenthesis r\rrparenthesis_{\theta,x=a} \mathbin{\#} \llparenthesis s\rrparenthesis_{\theta,x=a} - \llparenthesis r\rrparenthesis_{\theta,x=\mathbf{0}_n} \mathbin{\#} \llparenthesis s\rrparenthesis_{\theta,x=\mathbf{0}_n}\\
&= \llparenthesis r\rrparenthesis_{\gamma} \mathbin{\#} \llparenthesis s\rrparenthesis_{\delta,x=a} - \llparenthesis r\rrparenthesis_{\gamma} \mathbin{\#} \llparenthesis s\rrparenthesis_{\delta,x=\mathbf{0}_n}\\
(\text{Lemma 3.6}) &= \llparenthesis r\rrparenthesis_{\gamma} \mathbin{\#} (\llparenthesis s\rrparenthesis_{\delta,x=a} - \llparenthesis s\rrparenthesis_{\delta,x=\mathbf{0}_n}) - P_\perp(\llparenthesis r\rrparenthesis_{\gamma})\\
&= M \mathbin{\#} F^{s,x}_{\delta\vDash\Delta}(a) - M_2\\
&\quad (\text{with } \llparenthesis r\rrparenthesis_{\gamma} = M = M_1 \oplus M_2)
\end{aligned}
$$

    As $M \in (\llparenthesis A\rrparenthesis \otimes \llparenthesis B\rrparenthesis) \oplus \llparenthesis B\rrparenthesis$ is a positive matrix by inductive hypothesis, we have that:
    1. $M_1 \in \llparenthesis A\rrparenthesis \otimes \llparenthesis B\rrparenthesis$ is a positive matrix $\iff$ it is the characteristic matrix for some CPM $g : \llparenthesis A\rrparenthesis \to \llparenthesis B\rrparenthesis$.
    2. $M_2 \in \llparenthesis B\rrparenthesis$ is a positive matrix.

Then $M \# F^{s,x}_{\delta\vDash\Delta}(a) - M_2 = g(F^{s,x}_{\delta\vDash\Delta}(a)) + M_2 - M_2 = g(F^{s,x}_{\delta\vDash\Delta}(a))$ is a CPM by composition.

- Let $t = \mu_n y.r$. Then, by inversion, $\Gamma, y : B, x : A \vdash r : B$. Hence, by the induction hypothesis, we have that $F^{r,y}_{\theta,x=a\vDash\Gamma,x:A}$ and $F^{r,y}_{\theta,y=b\vDash\Gamma,y:B}$ are CPMs.
We need to prove that $F^{\mu_n y.r,x}_{\theta\vDash\Gamma}$ is a CPM.

$$\begin{aligned}F^{\mu_n y.r,x}_{\theta\vDash\Gamma}(a) &= (\!|\mu_n y.r|\!)_{\theta,x=a} - (\!|\mu_n y.r|\!)_{\theta,x=\mathbf{0}_m}\\ &= ((\!|\lambda y.r|\!)_{\theta,x=a} \#_n \mathbf{0}_m) - ((\!|\lambda y.r|\!)_{\theta,x=\mathbf{0}_m} \#_n \mathbf{0}_m)\\ \text{(Lemma 3.7)} &= ((\!|\lambda y.r|\!)_{\theta,x=a} - (\!|\lambda y.r|\!)_{\theta,x=\mathbf{0}_m}) \#_n \mathbf{0}_m\\ &= F^{\lambda y.r,x}_{\theta\vDash\Gamma}(a) \#_n \mathbf{0}_m\end{aligned}$$

With the same reasoning as in case $t = \lambda y.r$, we have that $F^{\lambda y.r,x}_{\theta\vDash\Gamma}(a)$ is a CPM. Therefore, $F^{\mu_n y.r,x}_{\theta\vDash\Gamma}$ is a CPM.
- Let $t = \bot$. Hence, we have to prove that $F^{\bot,x}_{\theta\vDash\Gamma}$ is a CPM.

$$F^{\bot,x}_{\theta\vDash\Gamma}(a) = (\!|\bot|\!)_{\theta,x=a} - (\!|\bot|\!)_{\theta,x=\mathbf{0}_n} = \mathbf{0}_m - \mathbf{0}_m = \mathbf{0}_m$$

Therefore, $F^{\bot,x}_{\theta\vDash\Gamma}$ is a CPM.
- Let $t = \rho^n$. Hence, we have to prove that $F^{\rho,x}_{\theta\vDash\Gamma}$ is a CPM.

$$F^{\rho,x}_{\theta\vDash\Gamma}(a) = (\!|\rho^n|\!)_{\theta,x=a} - (\!|\rho^n|\!)_{\theta,x=\mathbf{0}_n} = \rho - \rho = \mathbf{0}_{2^n}$$

Therefore, $F^{\rho,x}_{\theta\vDash\Gamma}$ is a CPM.
- Let $t = U^m r$. Then, by inversion, $B = n$ and $\Gamma, x : A \vdash r : n$. Hence, by the induction hypothesis $F^{r,x}_{\theta\vDash\Gamma}$ is a CPM.
We need to prove that $F^{Ur,x}_{\theta\vDash\Gamma}$ is a CPM.

$$\begin{aligned}F^{U^m r,x}_{\theta\vDash\Gamma}(a) &= (\!|U^m r|\!)_{\theta,x=a} - (\!|U^m r|\!)_{\theta,x=\mathbf{0}_m}\\ &= \overline{U}(\!|r|\!)_{\theta,x=a}\overline{U}^\dagger - \overline{U}(\!|r|\!)_{\theta,x=\mathbf{0}_m}\overline{U}^\dagger\\ &= \overline{U}\left((\!|r|\!)_{\theta,x=a} - (\!|r|\!)_{\theta,x=\mathbf{0}_m}\right)\overline{U}^\dagger\\ &= \overline{U}F^{r,x}_{\theta\vDash\Gamma}(a)\overline{U}^\dagger\end{aligned}$$

which is a CPM since $F^{r,x}_{\theta\vDash\Gamma}(a)$ is a CPM.
- Let $t = \pi^m r$. Then, by inversion, $B = n$ and $\Gamma, x : A \vdash r : n$. Hence, by the induction hypothesis $F^{r,x}_{\theta\vDash\Gamma}$ is a CPM.

$$\begin{aligned}F^{\pi^m r,x}_{\theta\vDash\Gamma}(a) &= (\!|\pi^m r|\!)_{\theta,x=a} - (\!|\pi^m r|\!)_{\theta,x=\mathbf{0}_k}\\ &= \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=a}\overline{|i\rangle\langle i|}^\dagger\right) - \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=\mathbf{0}_k}\overline{|i\rangle\langle i|}^\dagger\right)\\ &= \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=a}\overline{|i\rangle\langle i|}^\dagger - \overline{|i\rangle\langle i|}(\!|r|\!)_{\theta,x=\mathbf{0}_k}\overline{|i\rangle\langle i|}^\dagger\right)\\ &= \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}\left((\!|r|\!)_{\theta,x=a} - (\!|r|\!)_{\theta,x=\mathbf{0}_k}\right)\overline{|i\rangle\langle i|}^\dagger\right)\\ &= \bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}F^{r,x}_{\theta\vDash\Gamma}(a)\overline{|i\rangle\langle i|}^\dagger\right)\end{aligned}$$

which is a CPM since $F^{r,x}_{\theta\vDash\Gamma}(a)$ is a CPM.

- Let $t = r \otimes s$. Then, by inversion $B = n + m$ and there are two cases:
  - $\Gamma_1, x : A \vdash r : n$ and $\Gamma_2 \vdash s : m$, with $\Gamma = \Gamma_1, \Gamma_2$. Hence, by the induction hypothesis, $F^{r,x}_{\theta_1 \vDash \Gamma_1}$ is a CPM.
    We need to prove that $F^{r \otimes s, x}_{\theta \vDash \Gamma}$, with $\theta = \theta_1, \theta_2$, is a CPM.

$$\begin{aligned} F^{r\otimes s,x}_{\theta\vDash\Gamma}(a) &= (\!|r \otimes s|\!)_{\theta,x=a} - (\!|r \otimes s|\!)_{\theta,x=\mathbf{0}_k} \\ &= (\!|r|\!)_{\theta,x=a} \otimes (\!|s|\!)_{\theta,x=a} - (\!|r|\!)_{\theta,x=\mathbf{0}_k} \otimes (\!|s|\!)_{\theta,x=\mathbf{0}_k} \\ &= ((\!|r|\!)_{\theta_1,x=a} - (\!|r|\!)_{\theta_1,x=\mathbf{0}_k}) \otimes (\!|s|\!)_{\theta_2} \\ &= F^{r,x}_{\theta_1\vDash\Gamma_1}(a) \otimes (\!|s|\!)_{\theta_2} \end{aligned}$$

    A CPM tensor a positive maps is a CPM.
  - $\Gamma_1 \vdash r : n$ and $\Gamma_2, x : A \vdash s : m$. This case is analogous to the previous case.
- Let $t = \mathsf{letcase}^{\circ}\ y = r\ \mathsf{in}\ \{t_0, \dots t_{2^m-1}\}$. By inversion, for all i, $\Delta_i, y : n \vdash t_i : B$ and $\Xi \vdash r : (m, n)$, with $\Gamma, x : A = \Delta_0, \dots, \Delta_{2^m-1}, \Xi$.
  Cases:
  - $\Xi = \Xi', x = a$. Hence, by the induction hypothesis, $F^{r,x}_{\xi\vDash\Xi} = (\!|r|\!)_{\xi,x=a} - (\!|r|\!)_{\xi,x=\mathbf{0}_n}$ is a CPM.
    We need to prove that $F^{\mathsf{letcase}^{\circ}\ y=r\ \mathsf{in}\ \{t_0,\dots,t_{2^m-1}\},x}_{\theta\vDash\Gamma}$, with $\theta = \delta_1, \dots, \delta_{2^m-1}, \xi$, is a CPM.
    Let $(\!|r|\!)_{\theta,x=c} = \bigoplus_{i=0}^{2^m-1} \rho_{i(c)}$. Then,

$$\begin{aligned} &F^{\mathsf{letcase}^{\circ}\ y=r\ \mathsf{in}\ \{t_0,\dots,t_{2^m-1}\},x}_{\theta\vDash\Gamma}(a) \\ &= (\!|\mathsf{letcase}^{\circ}\ y = r\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}|\!)_{\theta,x=a} - (\!|\mathsf{letcase}^{\circ}\ y = r\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}|\!)_{\theta,x=\mathbf{0}_n} \\ &= \left(\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_{i(a)}\right) (\!|t_i|\!)_{\theta,x=a,y=\rho_{i(a)}}\right) - \left(\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_{i(\mathbf{0}_n)}\right) (\!|t_i|\!)_{\theta,x=\mathbf{0}_n,y=\rho_{i(\mathbf{0}_n)}}\right) \\ &= \sum_{i=0}^{2^m-1} \left(\mathsf{tr}\left(\rho_{i(a)}\right) (\!|t_i|\!)_{\delta_i,y=\rho_{i(a)}} - \mathsf{tr}\left(\rho_{i(\mathbf{0}_n)}\right) (\!|t_i|\!)_{\delta_i,y=\rho_{i(\mathbf{0}_n)}}\right) \\ &\sqsupseteq \sum_{i=0}^{2^m-1} \left(\mathsf{tr}\left(\rho_{i(a)}\right) \left((\!|t_i|\!)_{\delta_i,y=\rho_{i(a)}} - (\!|t_i|\!)_{\delta_i,y=\rho_{i(\mathbf{0}_n)}}\right)\right) \\ &\sqsupseteq \sum_{i=0}^{2^m-1} \left(\mathsf{tr}\left(\rho_{i(a)}\right) \left((\!|t_i|\!)_{\delta_i,y=\rho_{i(a)}} - (\!|t_i|\!)_{\delta_i,y=\mathbf{0}_n}\right)\right) \\ &= \sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_{i(a)}\right) F^{t_i,y}_{\delta_i\vDash\Gamma}(a) \end{aligned}$$

    Positive linear combination of CPMs is a CPM.
  - For some k, $\Delta_k = \Delta'_k, x = a$ Hence, by the induction hypothesis, $F^{t_k,x}_{\delta_k\vDash\Delta_k}$ is a CPM.
    We need to prove that $F^{\mathsf{letcase}^{\circ}\ y=r\ \mathsf{in}\ \{t_0,\dots,t_{2^m-1}\},x}_{\theta\vDash\Gamma}$, with $\theta = \delta_1, \dots, \delta_{2^m-1}, \xi$, is a CPM.
    Let $(\!|r|\!)_{\xi} = \bigoplus_{i=0}^{2^m-1} \rho_i$. Then,

$$\begin{aligned} &F^{\mathsf{letcase}^{\circ}\ y=r\ \mathsf{in}\ \{t_0,\dots,t_{2^m-1}\},x}_{\theta\vDash\Gamma}(a) \\ &= (\!|\mathsf{letcase}^{\circ}\ y = r\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}|\!)_{\theta,x=a} - (\!|\mathsf{letcase}^{\circ}\ y = r\ \mathsf{in}\ \{t_0, \dots, t_{2^m-1}\}|\!)_{\theta,x=\mathbf{0}_n} \\ &= \left(\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_i\right) (\!|t_i|\!)_{\theta,x=a,y=\rho_i}\right) - \left(\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho_i\right) (\!|t_i|\!)_{\theta,x=\mathbf{0}_n,y=\rho_i}\right) \\ &= (\!|t_k|\!)_{\delta_k,y=\rho_k} - (\!|t_k|\!)_{\delta_k,y=\rho_k} \end{aligned}$$

$$= F^{t_k,x}_{\delta_k \vDash \Delta_k}$$

- Let $t = \sum_i p_i t_i$. Then, by inversion, for all i, $\Gamma, x : A \vdash t_i : B$. Hence, by the induction hypothesis, $F^{t_i,x}_{\theta\vDash\Gamma}$ are CPMs.

  We need to prove that $F^{\sum_i p_i t_i, x}_{\theta\vDash\Gamma}$ is a CPM.

$$\begin{aligned}F^{\sum_i p_i t_i, x}_{\theta\vDash\Gamma}(a) &= (\!|\sum_i p_i t_i|\!)_{\theta,x=a} - (\!|\sum_i p_i t_i|\!)_{\theta,x=\mathbf{0}_n} \\ &= \sum_i p_i (\!|t_i|\!)_{\theta,x=a} - \sum_i p_i (\!|t_i|\!)_{\theta,x=\mathbf{0}_n} \\ &= \sum_i p_i \left((\!|t_i|\!)_{\theta,x=a} - (\!|t_i|\!)_{\theta,x=\mathbf{0}_n} \right) \\ &= \sum_i p_i F^{t_i,x}_{\theta\vDash\Gamma}(a)\end{aligned}$$

  which is a CPM since each $F^{t_i,x}_{\theta\vDash\Gamma}(a)$ are CPMs. ◀

B.6 Proof of Lemma 4.6

We first restate the following theorem from [10, Theorem 6.5].

▶ **Theorem B.2.** *Let $F : \mathbb{C}^{n\times n} \to \mathbb{C}^{m\times m}$ be a linear operator, and let $\chi_F \in \mathbb{C}^{nm\times nm}$ be its characteristic matrix.*

(a) F if of the form $F(A) = UAU^\dagger$, for some $U \in \mathbb{C}^{m\times n}$, if and only if χ_F is pure.
(b) The following are equivalent:
  (i) F is completely positive.
  (ii) χ_F is positive.
  (iii) F if of the form $F(A) = \sum_i U_i A U_i^\dagger$, for some finite sequence of matrices $U_1, \dots, U_k \in \mathbb{C}^{m\times n}$. ◀

▶ **Lemma 4.6** (Adequacy for abstractions). *Let $\Gamma, x : A \vdash t : B$ and $\theta \vDash \Gamma$, such that $(\!|t|\!)_{\theta,x=a}, (\!|t|\!)_{\theta,x=\mathbf{0}_{\dim(A)}} \in (\!|B|\!)$. Then $\overline{\chi}_{[a\mapsto(\!|t|\!)_{\theta,x=a}]} \in (\!|A \multimap B|\!)$.*

Proof. Using Lemma 4.1 we have that $a \mapsto (\!|t|\!)_{\theta,x=a} - (\!|t|\!)_{\theta,x=\perp}$ is a linear function. It also is completely positive by Lemma 4.5, therefore its characteristic matrix is positive by Theorem B.2.

Its characteristic matrix is the following:

$$M_t = \begin{pmatrix} (\!|t|\!)_{\theta,x=E_{11}} - (\!|t|\!)_{\theta,x=\perp} & \cdots & (\!|t|\!)_{\theta,x=E_{1n}} - (\!|t|\!)_{\theta,x=\perp} \\ \vdots & \ddots & \vdots \\ (\!|t|\!)_{\theta,x=E_{n1}} - (\!|t|\!)_{\theta,x=\perp} & \cdots & (\!|t|\!)_{\theta,x=E_{nn}} - (\!|t|\!)_{\theta,x=\perp} \end{pmatrix}$$

By Definition (see Section 3.3), we have that

$$\overline{\chi}_{[a\mapsto(\!|t|\!)_{\theta,x=a}]} = M_t \oplus (\!|t|\!)_{\theta,x=\perp}$$

Since $M_t \in (\!|A|\!) \otimes (\!|B|\!)$, we have $\overline{\chi}_{[a\mapsto(\!|t|\!)_{\theta,x=a}]} \in ((\!|A|\!) \otimes (\!|B|\!)) \oplus (\!|B|\!)$. As $(\!|t|\!)_{\theta,x=\perp}$ is in $(\!|B|\!)$, it is a positive matrix. Then $\overline{\chi}_{[a\mapsto(\!|t|\!)_{\theta,x=a}]}$ is a positive matrix because it is a coproduct between positive matrices, and since it is in $((\!|A|\!) \otimes (\!|B|\!)) \oplus (\!|B|\!)$, it belongs to $(\!|A \multimap B|\!)$. ◀

B.7 Proof of Lemma 4.7

▶ **Lemma 4.7** (Adequacy for arrow-type terms). *Let $\Gamma \vdash t : A \multimap B$ and $\theta \vDash \Gamma$. One of the following holds:*

- *There exist $t_1, \ldots, t_n$ and $p_1, \ldots, p_n$ such that $x : A \vdash t_i : B$, $p_i > 0$, $\sum_{i=1}^n p_i \leq 1$ and $(\!|t|\!)_\theta = \sum_{i=1}^n p_i (\!|\lambda x.t_i|\!)_\emptyset$.*
- *$(\!|t|\!)_\theta = \mathbf{0}_{\dim(A \multimap B)}$*

Proof. Let τ be a closure function such that $\tau \vDash \Gamma$ and $\tau \leftrightarrow \theta$. By substitution we have that $\vdash \tau(t) : A \multimap B$. Using Progress we have that either $\tau(t)$ is in $\mathsf{Val} \cup \{\bot\}$ or it rewrites, and by Lemma 4.3 we have that $(\!|t|\!)_\theta = (\!|\tau(t)|\!)_\emptyset$.

- If $\tau(t)$ is in $\mathsf{Val} \cup \{\bot\}$, by its type we have that either $\tau(t) = \sum_{i=1}^n p_i(\lambda x.t_i)$ for a variable x and terms t_i (such that $x : A \vdash t_i : B$, $0 < p_i \leq 1$ and $\sum_{i=1}^n p_i \leq 1$), or $\tau(t) = \bot$.
  In the first case we have:
  $$(\!|t|\!)_\theta = (\!|\tau(t)|\!)_\emptyset = (\!|\sum_i p_i(\lambda x.t_i)|\!)_\emptyset = \sum_i p_i (\!|\lambda x.t_i|\!)_\emptyset$$
  In the second case we have:
  $$(\!|t|\!)_\theta = (\!|\tau(t)|\!)_\emptyset = (\!|\bot|\!)_\emptyset = \mathbf{0}_{\dim(A \multimap B)}$$
- If $\tau(t) \longrightarrow r$, since $\vdash \tau(t) : A \multimap B$, by Subject reduction we have that $\vdash r : A \multimap B$. As $\tau(r) = r$ because r is a closed term, using Progress successively we have that there exist $r_1, \ldots, r_{n-1}$ closed terms and r_n in $\mathsf{Val} \cup \{\bot\}$, with $\vdash r_i : A \multimap B$, such that
  $$\tau(t) \longrightarrow r \longrightarrow r_1 \longrightarrow \cdots \longrightarrow r_n$$
  This holds because there are no infinite rewritings, by the strong normalisation property of the calculus. By Theorem 4.4 we have that:
  $$(\!|t|\!)_\theta = (\!|\tau(t)|\!)_\emptyset = (\!|r|\!)_\emptyset = (\!|r_1|\!)_\emptyset = \cdots = (\!|r_n|\!)_\emptyset$$
  As $\vdash r_n : A \multimap B$ and $r_n \in \mathsf{Val} \cup \{\bot\}$, we have the same results as in the previous item. ◀

B.8 Proof of Lemma 4.8

▶ **Lemma 4.8.** *Let $(\!|t_i|\!)_\theta \in (\!|A|\!)$ for $i \in \{1, \ldots, n\}$. Then for any $p_1, \ldots, p_n$ such that $0 < p_i \leq 1$ with $\sum_{i=1}^n p_i \leq 1$, we have $\sum_{i=1}^n p_i (\!|t_i|\!)_\theta \in (\!|A|\!)$.*

Proof. We prove more generally the following result: Let A be a type. For i in $\{1, \ldots, n\}$, let $a_i \in (\!|A|\!)$ and $0 < p_i \leq 1$ with $\sum_{i=1}^n p_i \leq 1$. Then $\sum_{i=1}^n p_i a_i \in (\!|A|\!)$.
We proceed by induction on types.

- If $A = n$, we have $(\!|A|\!) = \mathcal{D}_n$.
  Since $p_i > 0$ for all i and positive matrices form a vector space over $\mathbb{R}_{\geq 0}$, we have that $\sum_{i=1}^n p_i a_i$ is a positive matrix in $\mathbb{C}^{n \times n}$.
  By trace linearity, we have that $\mathsf{tr}\,(\sum_{i=1}^n p_i a_i) = \sum_{i=1}^n p_i \mathsf{tr}\,(a_i)$. This is bounded by $\sum_{i=1}^n p_i \leq 1$ because by hypothesis $a_i \in (\!|n|\!) = \mathcal{D}_n$.

- If $A = (m,n)$, we have $(\!|A|\!) = \{p \mid p \in \bigoplus_{i=1}^{2^m-1} \mathcal{D}_n \text{ y } \mathsf{tr}\,(p) \leq 1\}$.
  By hypothesis $a_i \in (\!|(m,n)|\!)$, so we can rewrite the a_i as $a_i = \bigoplus_{j=0}^{2^m-1} a_{ij}$, where $a_{ij} \in \mathcal{D}_n$. Then,

$$\sum_{i=1}^{n} p_i a_i = \sum_{i=1}^{n} p_i \left(\bigoplus_{j=0}^{2^m-1} a_{ij} \right) = \bigoplus_{j=0}^{2^m-1} \left(\sum_{i=1}^{n} p_i a_{ij} \right)$$

  By case $A = n$ we have that $\sum_{i=1}^n p_i a_{ij} \in \mathcal{D}_n$ for all $j \in \{0,\dots,2^m-1\}$, therefore $\sum_{i=1}^n p_i a_i \in \bigoplus_{i=0}^{2^m-1} \mathcal{D}_n$.
  By trace linearity, we have that $\mathsf{tr}\,(\sum_{i=1}^n p_i a_i) = \sum_{i=1}^n p_i \mathsf{tr}\,(a_i)$. By hypothesis $a_i \in (\!|(m,n)|\!)$, then this is bounded by $\sum_{i=1}^n p_i \leq 1$.
- If $A = B \multimap C$, we have $(\!|A|\!) = \{f \mid f \text{ positive in } ((\!|B|\!) \otimes (\!|C|\!)) \oplus (\!|C|\!)\}$.
  Since $\sum_{i=1}^n p_i a_i$ is a positive real combination of elements from $(\!|B \multimap C|\!)$, it is in $(\!|B \multimap C|\!)$. ◀

B.9 Proof of Theorem 4.9

We first give a lemma stating that function application preserves positivity. This implies that function application interpretation stays inside the domain.

▶ **Lemma B.3** ($\#$ preserves positivity)**.** *Let t be a term and θ be a valuation such that $(\!|\lambda x.t|\!)_\theta \in (\!|A \multimap B|\!)$, then for all a in $(\!|A|\!)$ we have that $(\!|t|\!)_{\theta,x=a} \in (\!|B|\!)$.*

Proof. By Lemma 4.2 we have $(\!|t|\!)_{\theta,x=a} = (\!|\lambda x.t|\!)_\theta \# a$. By hypothesis $(\!|\lambda x.t|\!)_\theta \in (\!|A \multimap B|\!)$, so we can write $(\!|\lambda x.t|\!)_\theta = M_1 \oplus M_2$ with $M_1 \in (\!|A|\!) \otimes (\!|B|\!)$ and $M_2 \in (\!|B|\!)$, both positive matrices. By definition of the $\#$ operator, $(M_1 \oplus M_2) \# a = M_1 @ a + M_2$. Since @ preserves positivity, therefore $M_1 @ a$ is in $(\!|B|\!)$ and so is the sum. ◀

We also give two auxiliary lemmas and a corollary, concerning trace bounds after projection of states. There are used in the measurement case of the proof of adequacy.

▶ **Lemma B.4.** *Let ρ be a positive matrix in $\mathbb{C}^{2^n \times 2^n}$. Then $\mathsf{tr}\left(\bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right)\right) = \mathsf{tr}\,(\rho)$ for all $m \leq n$.*

Proof. By trace linearity:

$$\mathsf{tr}\left(\bigoplus_{i=0}^{2^m-1}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right)\right) = \sum_{i=0}^{2^m-1} \mathsf{tr}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right)$$

By the trace cyclic property, this is equal to $\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho\overline{|i\rangle\langle i|}^\dagger\overline{|i\rangle\langle i|}\right)$. Since $\overline{|i\rangle\langle i|}$ is Hermitian and is also a projector, we have that $\overline{|i\rangle\langle i|}^\dagger\overline{|i\rangle\langle i|} = \overline{|i\rangle\langle i|}$, and the term is equal to $\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho\overline{|i\rangle\langle i|}\right)$. Since $\sum_{i=0}^{2^m-1} \overline{|i\rangle\langle i|} = \mathbf{1}_{2^n}$:

$$\sum_{i=0}^{2^m-1} \mathsf{tr}\left(\rho\overline{|i\rangle\langle i|}\right) = \mathsf{tr}\left(\sum_{i=0}^{2^m-1} \rho\overline{|i\rangle\langle i|}\right) = \mathsf{tr}\left(\rho\sum_{i=0}^{2^m-1} \overline{|i\rangle\langle i|}\right) = \mathsf{tr}\,(\rho)$$ ◀

▶ **Corollary B.5.** *Let ρ be a positive matrix in $\mathbb{C}^{2^n \times 2^n}$. For all $m \leq n$ and $0 \leq i \leq 2^m - 1$ we have that $\mathsf{tr}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right) \leq \mathsf{tr}\,(\rho)$.*

Proof. By bounding every sum term separately. ◀

▶ **Lemma B.6.** *Let ρ be a positive matrix in $\mathbb{C}^{2^n \times 2^n}$ such that $\mathsf{tr}(\rho) \leq 1$. Then for all $m \leq n$, for all $0 \leq i \leq 2^m - 1$ we have that $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ is positive and its trace is bounded by 1.*

Proof.
- $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ is Hermitian: we have $(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger)^\dagger = \overline{|i\rangle\langle i|}\rho^\dagger\overline{|i\rangle\langle i|}^\dagger$. This is equal to $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ because ρ es is Hermitian by hypothesis.
- $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ is semidefinite positive: multiplying to both sides by a vector u in $\mathbb{C}^{2^n}$ we have that $u^\dagger\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger u = (u^\dagger\overline{|i\rangle\langle i|})\rho(\overline{|i\rangle\langle i|}^\dagger u)$. Let $v = \overline{|i\rangle\langle i|}^\dagger u \in \mathbb{C}^{2^n}$, we have that $v^\dagger = (\overline{|i\rangle\langle i|}^\dagger u)^\dagger = u^\dagger\overline{|i\rangle\langle i|}$. Therefore $(u^\dagger\overline{|i\rangle\langle i|})\rho(\overline{|i\rangle\langle i|}^\dagger u) = v^\dagger\rho v \geq 0$ because ρ is semidefinite positive.
- The trace of $\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger$ is bounded by 1: by Corollary B.5 we have that $\mathsf{tr}\left(\overline{|i\rangle\langle i|}\rho\overline{|i\rangle\langle i|}^\dagger\right) \leq \mathsf{tr}(\rho) \leq 1$. ◀

▶ **Theorem 4.9** (Adequacy). *Let $\Gamma \vdash t : A$ and $\theta \vDash \Gamma$, then $(\!|t|\!)_\theta \in (\!|A|\!)$.*

Proof. By induction on the typing rules.

- $$\frac{}{\Gamma, x : A \vdash x : A}\ \mathsf{ax}$$
  In this case we have $(\!|x|\!)_\theta = \theta(x)$ by definition. By hypothesis we have that $\theta \vDash \Gamma, x : A$, therefore $\theta(x) = (\!|x|\!)_\theta \in (\!|A|\!)$.
- $$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \multimap B}\ \multimap_i$$
  By the inductive hypothesis we have that for all θ' such that $\theta' \vDash \Gamma, x : A$, $(\!|t|\!)_{\theta'} \in (\!|B|\!)$. Let $a \in (\!|A|\!)$, then $\theta' = \theta \cup \{x = a\} \vDash \Gamma, x : A$, therefore $(\!|t|\!)_{\theta, x=a} \in (\!|B|\!)$. Also, $\mathbf{0}_{\dim(A)} \in (\!|A|\!)$, and then $(\!|t|\!)_{\theta, x=\perp} \in (\!|B|\!)$. By definition we have that $(\!|\lambda x.t|\!)_\theta = \overline{\chi}_{[a \mapsto (\!|t|\!)_{\theta, x=a}]}$ and by Lemma 4.6 this is in $(\!|A \multimap B|\!)$.
- $$\frac{\Gamma \vdash t : A \multimap B \quad \Delta \vdash r : A}{\Gamma, \Delta \vdash tr : B}\ \multimap_e$$
  Since $\theta \vDash \Gamma, \Delta$, we have that $\theta \vDash \Gamma$ and $\theta \vDash \Delta$. By the inductive hypothesis, $(\!|t|\!)_\theta \in (\!|A \multimap B|\!)$ and $(\!|r|\!)_\theta \in (\!|A|\!)$. By Lemma 4.7 we have that either that for some $n \in \mathbb{N}$ there exist n terms $t_1, \ldots, t_n$ and n real numbers $p_1, \ldots, p_n$ such that $(\!|t|\!)_\theta = \sum_{i=1}^n p_i (\!|\lambda x.t_i|\!)_\emptyset$ (with $x : A \vdash t_i : B$, $0 < p_i \leq 1$ y $\sum_{i=1}^n p_i \leq 1$), or else $(\!|t|\!)_\theta = \mathbf{0}_{\dim(A \multimap B)}$.
  By definition, $(\!|tr|\!)_\theta = (\!|t|\!)_\theta \ \#\ (\!|r|\!)_\theta$.
  - In the first case we have:
    $$(\!|tr|\!)_\theta = \left(\sum_{i=1}^n p_i (\!|\lambda x.t_i|\!)_\emptyset\right) \ \#\ (\!|r|\!)_\theta = \sum_{i=1}^n p_i ((\!|\lambda x.t_i|\!)_\emptyset \ \#\ (\!|r|\!)_\theta)$$
    By Lemma 4.2 this is equal to $\sum_{i=1}^n p_i (\!|t_i|\!)_{x=(\!|r|\!)_\theta}$. By Lemma B.3, $(\!|t_i|\!)_{x=(\!|r|\!)_\theta} \in (\!|B|\!)$ for all i. By Lemma 4.8 this linear combination is in $(\!|B|\!)$.
  - In the second case we have:
    $$(\!|tr|\!)_\theta = \mathbf{0}_{\dim(A \multimap B)} \ \#\ (\!|r|\!)_\theta = \mathbf{0}_{\dim(B)} \in (\!|B|\!)$$
    This holds because according to the definition for the $\#$ operator, $\mathbf{0}_{\dim(A \multimap B)}$ is the constant function $a \mapsto \mathbf{0}_{\dim(B)}$.
- $$\frac{\Gamma, f : A \vdash t : A}{\Gamma \vdash \mu_n f.t : A}\ \mu$$

Let Γ be a typing context and let θ be a valuation such that $\Gamma \vdash \mu_n f.t : A$ and $\theta \vDash \Gamma$. Then by inversion we have that $\Gamma, f : A \vdash t : A$. Using rule $\multimap_i$ we have $\Gamma \vdash \lambda f.t : A \multimap A$. By the adequacy case for rule $\multimap_i$, we have that $(\!|\lambda f.t|\!)_\theta \in (\!|A \multimap A|\!)$.
We want to show that $(\!|\mu_n f.t|\!)_\theta = (\!|\lambda f.t|\!)_\theta \#_n \mathbf{0}_{\dim(A)}$ is in $(\!|A|\!)$. By induction on n:

- Base case: $(\!|\lambda f.t|\!)_\theta \#_0 \mathbf{0}_{\dim(A)} = \mathbf{0}_{\dim(A)} \in (\!|A|\!)$ by definition.
- $(\!|\lambda f.t|\!)_\theta \#_{n+1} \mathbf{0}_{\dim(A)} = (\!|\lambda f.t|\!)_\theta \# ((\!|\lambda f.t|\!)_\theta \#_n \mathbf{0}_{\dim(A)})$. By the inductive hypothesis we have that $(\!|\lambda f.t|\!)_\theta \#_n \mathbf{0}_{\dim(A)} \in (\!|A|\!)$. Since $(\!|\lambda f.t|\!)_\theta$ is in $(\!|A \multimap A|\!)$ we have that $(\!|\lambda f.t|\!)_\theta \#_{n+1} \mathbf{0}_{\dim(A)}$ is in $(\!|A|\!)$ by Lemma B.3.

- $\dfrac{}{\Gamma \vdash \bot : A}\ \bot$
  By definition we have that $(\!|\bot|\!)_\theta = \mathbf{0}_{\dim(A)}$. The null matrix is Hermitian, positive semidefinite and its trace is bounded by 1, so $\mathbf{0}_{\dim(A)} \in (\!|A|\!)$ for all type A.
- $\dfrac{}{\Gamma \vdash \rho^n : n}\ \mathsf{ax}_\rho$
  For all θ, in particular such that $\theta \vDash \Gamma$, we have $(\!|\rho^n|\!)_\theta = \rho \in \mathcal{D}_n = (\!|n|\!)$.
- $\dfrac{\Gamma \vdash t : n}{\Gamma \vdash U^m t : n}\ \mathsf{u}_i$
  By the inductive hypothesis we have that for all θ' such that $\theta' \vDash \Gamma$, $(\!|t|\!)_{\theta'} \in (\!|n|\!) = \mathcal{D}_n$. Since $\theta \vDash \Gamma$, then $(\!|t|\!)_\theta \in \mathcal{D}_n$. By definition we have that $(\!|U^m t|\!)_\theta = \overline{U}(\!|t|\!)_\theta \overline{U}^\dagger$. $\overline{U}$ is a unitary matrix, and so this product is in $\mathcal{D}_n$.
- $\dfrac{\Gamma \vdash t : n}{\Gamma \vdash \pi^m t : (m,n)}\ \mathsf{m}_i$
  By the inductive hypothesis we have that for all θ' such that $\theta' \vDash \Gamma$, $(\!|t|\!)_{\theta'} \in (\!|n|\!) = \mathcal{D}_n$. Since $\theta \vDash \Gamma$, then $(\!|t|\!)_\theta \in \mathcal{D}_n$. By definition we have that $(\!|\pi^m t|\!)_\theta = \bigoplus_{i=0}^{2^m-1}(\overline{|i\rangle\langle i|}(\!|t|\!)_\theta \overline{|i\rangle\langle i|}^\dagger)$. As $(\!|t|\!)_\theta$ is in $\mathcal{D}_n$, by Lemma B.6 $\overline{|i\rangle\langle i|}(\!|t|\!)_\theta \overline{|i\rangle\langle i|}^\dagger$ it is in $\mathcal{D}_n$. By Lemma B.4, we have that $\mathsf{tr}\,((\!|\pi^m t|\!)_\theta) = \mathsf{tr}\left(\bigoplus_{i=0}^{2^m-1} \overline{|i\rangle\langle i|}(\!|t|\!)_\theta \overline{|i\rangle\langle i|}^\dagger\right) = \mathsf{tr}\,((\!|t|\!)_\theta)$, and this is bounded by 1 by definition of $\mathcal{D}_n$.
  Therefore $(\!|\pi^m r|\!)_\theta \in \bigoplus_{i=0}^{2^m-1} \mathcal{D}_n = (\!|(m,n)|\!)$.
- $\dfrac{\Gamma \vdash t : n \quad \Delta \vdash r : m}{\Gamma, \Delta \vdash t \otimes r : n+m}\ \otimes$
  By the inductive hypothesis we have that for all θ' such that $\theta' \vDash \Gamma$, $(\!|t|\!)_{\theta'} \in (\!|n|\!) = \mathcal{D}_n$; and for all θ'' such that $\theta'' \vDash \Delta$, $(\!|t|\!)_{\theta''} \in (\!|m|\!) = \mathcal{D}_m$.
  By hypothesis we have that $\theta \vDash \Gamma, \Delta$, then $\theta \vDash \Gamma$ and $\theta \vDash \Delta$. Thus, $(\!|t|\!)_\theta \in \mathcal{D}_n$ and $(\!|r|\!)_\theta \in \mathcal{D}_m$.
  Then we have that $(\!|r \otimes s|\!)_\theta = (\!|r|\!)_\theta \otimes (\!|s|\!)_\theta \in \mathcal{D}_{n+m} = (\!|n+m|\!)$ because tensor product arity is given by $\otimes \colon \mathcal{D}_n \times \mathcal{D}_m \to \mathcal{D}_{n+m}$.
- $\dfrac{\Gamma \vdash t_1 : A \quad \ldots \quad \Gamma \vdash t_n : A \quad \sum_{i=1}^n p_i \leq 1 \quad \ell(A) \neq (m,n)}{\Gamma \vdash \sum_{i=1}^n p_i t_i : A}\ +$
  By the inductive hypothesis we have that for all θ' such that $\theta' \vDash \Gamma$, $(\!|t_i|\!)_{\theta'} \in (\!|A|\!)$ for all i in $\{1, \ldots, n\}$.
  By hypothesis $\theta \vdash \Gamma$, then $(\!|t_i|\!)_\theta \in (\!|A|\!)$ for all i in $\{1, \ldots, n\}$.
  By definition we have that $(\!|\sum_{i=1}^n p_i t_i|\!)_\theta = \sum_{i=1}^n p_i (\!|t_i|\!)_\theta$, and by Lemma 4.8 this is in $(\!|A|\!)$.
- $\dfrac{\Delta_0, x : n \vdash t_0 : A \quad \ldots \quad \Delta_{2^m-1}, x : n \vdash t_{2^m-1} : A \quad \Gamma \vdash r : (m,n) \quad \ell(A) \neq (m',n')}{\Delta_0, \ldots, \Delta_{2^m-1}, \Gamma \vdash \mathsf{letcase}^\circ\ x = r\ \mathsf{in}\ \{t_0, \ldots, t_{2^m-1}\} : A}\ \mathsf{m}_e$
  By the inductive hypothesis we have that for all θ' such that $\theta' \vDash \Gamma$ we have that $(\!|r|\!)_{\theta'} \in (\!|(m,n)|\!)$. $\theta \vDash \Delta_0, \ldots, \Delta_{2^m-1}, \Gamma$, then $\theta \vDash \Gamma$ and so $(\!|r|\!)_\theta \in (\!|(m,n)|\!)$.
  Also by the inductive hypothesis, for all i in $\{0, \ldots, 2^m - 1\}$, for all θ' such that $\theta' \vDash \Delta_i, x : n$ we have that $(\!|t_i|\!)_{\theta'} \in (\!|A|\!)$. Since $\theta \vDash \Delta_0, \ldots, \Delta_{2^m-1}, \Gamma$, in particular $\theta \vDash \Delta_i$ for

all i and $\theta \cup \{x = \rho\} \vDash \Delta_i, x : n$ for all $\rho \in \mathcal{D}_n$. Therefore $(\!|t_i|\!)_{\theta,x=\rho} \in (\!|A|\!)$ for all $\rho \in \mathcal{D}_n$ and all $i \in \{0, \ldots, 2^m - 1\}$.
By definition we have:

$$(\!|\mathsf{letcase}^\circ\ x = r \ \mathsf{in}\ \{t_0, \ldots, t_{2^m-1}\}|\!)_\theta = \sum_{i=0}^{2^m-1} \mathsf{tr}\,(\rho_i)\,(\!|t_i|\!)_{\theta,x=\rho'_i}$$

where $(\!|r|\!)_\theta = \bigoplus_{i=0}^{2^m-1} \rho_i \in (\!|(m,n)|\!)$ and

$$\rho'_i = \begin{cases} \dfrac{\rho_i}{\mathsf{tr}\,(\rho_i)} & \text{if } \mathsf{tr}\,(\rho_i) \neq 0 \\ \rho_i & \text{if } \mathsf{tr}\,(\rho_i) = 0 \end{cases}$$

$\rho'_i \in \mathcal{D}_n$ because $\rho_i \in \mathcal{D}_n$, therefore by inductive hypothesis $(\!|t_i|\!)_{\theta,x=\rho'_i} \in (\!|A|\!)$ for all i in $\{0, \ldots, 2^m - 1\}$.
Since $\rho_i \in \mathcal{D}_n$ for all i, we have $0 \leq \mathsf{tr}\,(\rho_i) \leq 1$. Also, as $(\!|r|\!)_\theta \in (\!|(m,n)|\!)$, we have $\mathsf{tr}\,((\!|r|\!)_\theta) \leq 1$. Therefore,

$$\mathsf{tr}\,((\!|r|\!)_\theta) = \mathsf{tr}\left(\bigoplus_{i=0}^{2^m-1} \rho_i\right) = \sum_{i=0}^{2^m-1} \mathsf{tr}\,(\rho_i) \leq 1$$

By Lemma 4.8, $\sum_{i=0}^{2^m-1} \mathsf{tr}\,(\rho_i)\,(\!|t_i|\!)_{\theta,x=\rho'_i} \in (\!|A|\!)$. ◀

C Proofs of Section 4.2

C.1 Proof of Theorem 4.11

▶ **Theorem 4.11.** *Let* $\vdash t\colon A$, *then* $\mathsf{tr}\,((\!|t|\!)_\emptyset) \leq N_A$.

Proof. By induction on types.
By Theorem 4.9 we have that $(\!|t|\!)_\emptyset \in (\!|A|\!)$ for every type A.

- If $A = n$ or $A = (m,n)$, we have that $\mathsf{tr}\,((\!|t|\!)_\emptyset) \leq 1$ by definition of $(\!|n|\!)$ and $(\!|(m,n)|\!)$.
- If $A = B \multimap C$, by Lemma 4.7 there are two possibilities:
  - For $n \in \mathbb{N}$, there are n terms $t_1, \ldots, t_n$ and n real numbers $p_1, \ldots, p_n$ such that $x : B \vdash t_i : C$, $0 < p_i \leq 1$, $\sum_{i=1}^n p_i \leq 1$ and $(\!|t|\!)_\emptyset = \sum_{i=1}^n p_i (\!|\lambda x.t_i|\!)_\emptyset$. By trace linearity:

$$\begin{aligned}
\mathsf{tr}\,((\!|t|\!)_\emptyset) &= \mathsf{tr}\left(\sum_{i=1}^n p_i (\!|\lambda x.t_i|\!)_\emptyset\right) \\
&= \sum_{i=1}^n p_i \mathsf{tr}\,((\!|\lambda x.t_i|\!)_\emptyset) \\
&= \sum_{i=1}^n p_i \left(\sum_{j=1}^{\dim(B)} \mathsf{tr}\left((\!|t_i|\!)_{x=E^B_{ii}} - (\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}}\right) + \mathsf{tr}\left((\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}}\right)\right) \\
&= \sum_{i=1}^n p_i \left(\sum_{j=1}^{\dim(B)} \mathsf{tr}\left((\!|t_i|\!)_{x=E^B_{ii}}\right) - \sum_{j=1}^{\dim(B)} \mathsf{tr}\,((\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}}) + \mathsf{tr}\,((\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}})\right)
\end{aligned}$$

By Lemma B.3, both $(\!|t_i|\!)_{x=E^B_{ii}}$ and $(\!|t_i|\!)_{x=\mathbf{0}_{\dim(B)}}$ are in $(\!|C|\!)$, for all i. Therefore, and by the inductive hypothesis, all those terms are positive and bounded by N_C. Then,

$$\mathsf{tr}\left((\!|t|\!)_\emptyset\right) \leq \sum_{i=1}^{n} p_i \left(\sum_{j=1}^{\dim(B)} N_C + N_C\right) = \sum_{i=1}^{n} p_i (N_{B\multimap C}) \leq N_{B\multimap C}$$

The last inequality holds because of the bound on the probability sum.

- $(\!|t|\!)_\emptyset = \mathbf{0}_{\dim(B\multimap C)}$
  This case is trivial: $\mathsf{tr}\left((\!|t|\!)_\emptyset\right) = \mathsf{tr}\left(\mathbf{0}_{\dim(B\multimap C)}\right) = 0 \leq N_{B\multimap C}$ ◀

C.2 Proof of Lemma 4.13

▶ **Lemma 4.13.** *Let $\Gamma, x : A \vdash t : A$ and $\theta \vDash \Gamma$. Then for all $a, b \in (\!|A|\!)$, if $a \sqsubseteq b$ we have $(\!|\lambda x.t|\!)_\theta \# a \sqsubseteq (\!|\lambda x.t|\!)_\theta \# b$.*

Proof. By Lemma 4.5 and Lemma 4.1, we have that the following function is linear and completely positive, where $n = \dim(A)$.

$$f(c) = (\!|t|\!)_{\theta,x=c} - (\!|t|\!)_{\theta,x=\mathbf{0}_n}$$

Since $a \sqsubseteq b$ by hypothesis, $b - a$ is a positive matrix and since f is completely positive, $f(b-a)$ is also a positive matrix. In addition f is linear, then $f(b-a) = f(b) - f(a)$ is a positive matrix.

$$\begin{aligned} f(b-a) =& (\!|t|\!)_{\theta,x=b} - (\!|t|\!)_{\theta,x=\mathbf{0}_n} - \left((\!|t|\!)_{\theta,x=a} - (\!|t|\!)_{\theta,x=\mathbf{0}_n}\right) \\ =& (\!|t|\!)_{\theta,x=b} - (\!|t|\!)_{\theta,x=a} \end{aligned}$$

By Lemma 4.2 we have that $(\!|t|\!)_{\theta,x=a} = (\!|\lambda x.t|\!)_\theta \# a$ and $(\!|t|\!)_{\theta,x=b} = (\!|\lambda x.t|\!)_\theta \# b$. Rewriting we have:

$$f(b-a) = (\!|\lambda x.t|\!)_\theta \# b - (\!|\lambda x.t|\!)_\theta \# a$$

Therefore $(\!|\lambda x.t|\!)_\theta \# b - (\!|\lambda x.t|\!)_\theta \# a$ is a positive matrix, and this implies

$$(\!|\lambda x.t|\!)_\theta \# a \sqsubseteq (\!|\lambda x.t|\!)_\theta \# b$$ ◀

C.3 Proof of Lemma 4.14

▶ **Lemma 4.14.** *Let $\overline{\chi} \in \mathbb{C}^{nm\times nm} \oplus \mathbb{C}^{m\times m}$ and let (P_n) be an increasing sequence of positive matrices in $\mathbb{C}^{n\times n}$ such that $\lim_{n\to\infty} P_n = P$. Then, $\overline{\chi}$ is monotone and $\lim_{n\to\infty} \overline{\chi} \# P_n = \overline{\chi} \# P$.*

Proof. By Lemma 4.13 we have that interpretations of abstraction terms in this calculus are monotone with respect to the Löwner order.

Remark that for all $1 \leq i, j \leq n, \lim_{n\to\infty}(P_n)_{ij} = P_{ij}$, where $(P_n)_{ij}$ is a sequence in $\mathbb{C}$. Let L_{ij}, K in $\mathbb{C}^{m\times m}$ and $\{E^n_{ij}\}$ the canonical basis for $\mathbb{C}^{n\times n}$ such that:

$$\overline{\chi} = \left(\sum_{i=1}^{n}\sum_{j=1}^{n} E^n_{ij} \otimes L_{ij}\right) \oplus K$$

Since the first term of this matrix acts linearly on the argument by Lemma 3.6, it is linear in every matrix element. Therefore application is continuous on every matrix element:

$$\lim_{n\to\infty} \overline{\chi} \# P_n = \lim_{n\to\infty} \sum_{i=1}^{n}\sum_{j=1}^{n} (P_n)_{ij} L_{ij} + K = \sum_{i=1}^{n}\sum_{j=1}^{n} P_{ij} L_{ij} + K = \overline{\chi} \# P$$ ◀

C.4 Proof of Lemma 4.15

We need first the following technical lemma.

▶ **Lemma C.1.** *For all positive M in $\mathbb{C}^{n\times n}$ and u in $\mathbb{C}^n$ we have that $u^\dagger M u \leq \mathsf{tr}\,(M)\,\|u\|^2$.*

Proof. Since M is positive, it is diagonalisable and can be written as $P^{-1}DP$ where $D \in \mathbb{C}^{n\times n}$ is diagonal and $P \in \mathbb{C}^{n\times n}$ is unitary.

Therefore we have that

$$u^\dagger M u = u^\dagger (P^{-1}DP)u = (u^\dagger P^{-1})D(Pu) = v^\dagger D v$$

Defining $v = Pu \in \mathbb{C}^n$. Let $v_i \in \mathbb{C}$ be v's elements:

$$u^\dagger M u = \sum_i v_i^2 d_{ii} \leq \left(\sum_i v_i^2\right)\left(\sum_i d_{ii}\right) = \mathsf{tr}\,(D)\,\|v\|^2 = \mathsf{tr}\,(M)\,\|u\|^2$$

This inequality holds because $d_{ii} \geq 0$ for all i since these are M's eigenvalues, that is a positive matrix. ◀

▶ **Lemma 4.15.** *For any type A, $(\mathfrak{D}_A, \sqsubseteq)$ is a complete partial order.*

Proof. We follow the structure of the proof at [10, Proposition 3.6].

We want to prove that in this set, the increasing sequences with respect to the Löwner order have a least upper bound.

$\mathfrak{D}_A$ is a subset of positive matrices in $\mathbb{C}^{2^n\times 2^n}$. Let M_1 and M_2 be matrices in $\mathfrak{D}_A$.

By definition $M_1 \sqsubseteq M_2$ if and only if $M_2 - M_1$ is a positive matrix, and this happens if and only if $u^\dagger(M_2 - M_1)u \geq 0$ for all u in $\mathbb{C}^{2^n}$. Therefore $M_1 \sqsubseteq M_2$ if and only if $u^\dagger M_1 u \leq u^\dagger M_2 u$ for all u in $\mathbb{C}^{2^n}$.

Thus, for all increasing sequences in $\mathfrak{D}_A$:

$$M_1 \sqsubseteq M_2 \sqsubseteq \ldots \sqsubseteq M_n \sqsubseteq \ldots$$

there is a corresponding increasing sequence in $\mathbb{R}_{\geq 0}$ for all u in $\mathbb{C}^{2^n}$:

$$u^\dagger M_1 u \leq u^\dagger M_2 u \leq \cdots \leq u^\dagger M_n u \leq \ldots$$

By Lemma C.1 and Corollary 4.11 we have that every element of the increasing sequence $\{u^\dagger M_n u\}$ are bounded by $N_A\,\|u\|^2$, since M_n matrices are in $\mathfrak{D}_A$. Any bounded increasing sequence in $\mathbb{R}$ has a least upper bound. Therefore the corresponding sequence $\{M_n\}$ in $\mathfrak{D}_A$ also has a least upper bound, and, by trace continuity, it is bounded by N_A. Hence, it is in $\mathfrak{D}_A$. ◀